Electronics for Resonant Sensors

by

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Kenneth Edward Wojciechowski
Abstract

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Resonant force or displacement sensing based on observing the change in resonant frequency is attractive because of its relative insensitivity to 1/f noise, high resolution and bandwidth, and “quasi-digital” output. Applications include inertial and strain sensors, biosensors based on mass-loading, and atomic force microscopy. The main contributions of this dissertation are

- Design of low-noise high fidelity MEMS resonators for sensing,
- Electronic oscillator circuits for these resonators, and
- Electronic demodulation of the frequency modulated output and conversion to a digital representation.

The proposed solutions are verified in a resonant MEMS strain sensor.
In resonant sensors, the purity of the oscillation signal determines the achievable resolution. This requires minimizing noise, which is usually attempted by maximizing the Q (quality factor) of the MEMS resonator. It is shown, however, that there exists a tradeoff between noise that is close to the carrier and noise far from the carrier. While the former decreases with Q as expected, far from carrier noise worsens when Q is increased. Especially in applications demanding relatively high bandwidth the optimal Q can be well below 100.

Because of the relatively weak interaction between the mechanical and electrical domains of MEMS resonators with electrostatic interfaces, these devices exhibit large series resistance, often in the mega-Ohm range. Low Q designs exacerbate this problem. In many setups the resulting small motional current is completely swamped by capacitive feedthrough, preventing oscillation with typical oscillator circuits. The proposed time variant square wave drive oscillator (SWO) overcomes the problem by separating motional and feedthrough current in the time domain. Reliable oscillation has been demonstrated for resonators with motional resistance in excess of $100\text{M} \Omega$ and orders of magnitude larger feedthrough than acceptable with traditional oscillator circuits.

The output from a resonant sensor is a frequency modulated sine-wave that must be converted to a digital representation. Owing to the typically small modulation index and moderate to high bandwidth requirements, simple solutions such as frequency counting and conventional PLLs cannot easily be used for this purpose. A new type of sigma-delta PLL ($\Sigma\Delta$PLL) addresses these challenges and combines both demodulation and digitization into a single step.
A prototype resonant strain sensor measurement system with a SWO oscillator and a ΣΔPLL was implemented with surface mount components on a PC board. With the SWO oscillator, we obtained a phase noise floor of –120 dBC/Hz. This is the best noise performance obtained to date for resonant sensors in this resonant frequency range. This confirms the conclusions from the model that predicted an improved phase noise floor as the Q is brought closer to its optimum value. The ΣΔPLL achieved a 1Hz frequency resolution (21 ps period resolution) in a 10kHz bandwidth and the overall strain sensor measurement system achieved a 33 nε resolution in 10kHz.

Bernhard E. Boser, Chair
To my family.
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CHAPTER 1:
INTRODUCTION

1.1 Introduction and Motivation

MEMS resonant sensors have been demonstrated in a wide range of applications such as atomic force microscopy, strain measurement and inertial navigation [1, 2, 3]. In each of these applications the resonant frequency is a function of the quantity to be measured. For instance the measurand may cause changes in the sensor stiffness or its mass resulting in a resonant frequency shift. The goal of this work is to implement a high resolution (0.1 \( \mu \varepsilon \) in 10kHz) resonant strain sensor to measure very small strain fields in automobile suspension systems for vehicle traction control.

To perform this measurement, electronics are used to interrogate the sensor to determine its resonant frequency. The electronics that perform this function are collectively referred to as a resonant sensor measurement system. This work concentrates on resonant sensor measurement systems that are composed of two components. The first component is a resonant sensor combined with electronics to create an oscillator, referred to as a resonant sensor oscillator. This oscillator output is equal to the resonant frequency of the sensor and hence its output is frequency modulated (FM) by the measurand [4]. This modulated frequency is measured and/or demodulated by the second component of the measurement system to output a signal proportional to the measurand.
Unfortunately, the output frequency of the resonant sensor oscillator is corrupted by noise from both the resonator and the electronics. This noise determines the minimum detectable change in frequency (frequency resolution) and hence the minimum detectable strain, force or mass (sensor resolution). Therefore it is invaluable to have a good understanding of how the properties of both the resonant sensor and electronics in the oscillator affect frequency noise.

One approach to determine oscillator frequency noise is to measure or calculate its phase noise. The frequency noise can then be derived from this phase noise. This method has been used in the past [1, 5] to estimate the resolution of resonant sensors. In [1, 5] phase noise was reported to be inversely proportional to both Q and the drive level of the resonator. Typically drive level corresponds to peak deflection, $x_b$, of the resonator at the oscillator fundamental frequency (typically the sensor resonant frequency). If phase noise is indeed inversely proportional to both Q and $x_b$, then resolution is limited by the physical limits of Q and $x_b$. However, a detailed analysis shows that a more complex relationship exists between phase noise, resonator drive level and Q.

This research presents a detailed model for phase noise (and hence resolution) that reveals a more complicated relationship between phase noise, Q and $x_b$. This model shows that, depending on the desired sensor bandwidth, there are values for both Q and $x_b$ that optimize resolution. In fact, the optimum Q will be shown to be proportional to the resonant frequency, $f_r$, and inversely proportional to the sensor bandwidth, BW. Hence, the phase noise model provides an estimate for the optimum resolution attainable with a resonant sensor.
Finally, an example design of a MEMS resonant strain sensor is presented based on a detailed model. This example demonstrates that the sensor performance improves by setting Q close to its optimum value improves sensor performance. Previous work had suggested that sensor performance could be arbitrarily improved by increasing sensor Q [1, 5].

For resonant sensors with frequencies in the 100’s of kilohertz with sensor bandwidths ranging from 100Hz to 1kHz, the optimum Q can be quite low (ranging from 100 to 1000). Operation at low Q can introduce complications in the oscillator design. Designing a low Q oscillator requires overcoming deleterious effects of parasitic feed-through capacitance. To solve this problem, a time-variant square wave drive oscillator (SWO) was developed that enabled oscillation of low Q sensors in the presence of large feed through capacitance.

The other challenge in this resonant sensor measurement system is to develop a high-resolution method to measure and digitize the output frequency of these oscillators. Typically a phase locked-loop (PLL) is used to demodulate the oscillator output [5]. Implementing this system with PLLs would require a two-step scheme that involves demodulation, performed by PLL, followed by a digitization step done by an A/D converter. Additionally, voltage controlled oscillators (VCOs) used in the PLL feedback loop adversely affect the performance of the measurement system.

We developed a sigma delta PLL (ΣΔPLL) to implement the frequency measurement system. This combines the demodulation and the digitization into a single step to simplify the design of the measurement system. In addition it takes advantage of oversampling
allowing simplification of the A/D converter design. The ΣΔPLL uses a digital counter instead of the VCO. The counter accuracy is dependent on the frequency reference used to drive it. With a high accuracy reference, such as an external crystal oscillator, the variation in the system due the counter is much lower than the variation introduced by a VCO.

This new resonant sensor measurement system with a SWO oscillator and a ΣΔPLL was implemented with surface mount components on a PC board to demonstrate the feasibility of these concepts. With this SWO oscillator, we obtained a phase noise floor of –120 dBc/Hz. This is the best noise performance obtained to date for resonant sensors in this resonant frequency range. This confirmed the conclusions from the model that predicted an improved phase noise floor as the Q is reduced. The ΣΔPLL implementation achieved a 1Hz frequency resolution (21 ps period resolution) in a 10kHz bandwidth. This overall measurement system was used to implement a prototype strain sensor and achieved a 33 nε resolution in 10kHz.

1.2 Chapter Organization

This thesis is organized as follows:

Chapter 2: Details the design of double-ended tuning fork resonant sensors. It provides the basis for the development of the sensors electrical equivalent model. In addition it discusses some of the non-ideal resonator characteristics such as nonlinearity and, temperature variation.
1.2 Chapter Organization

Chapter 3: Discusses the three different methods for detection of the sensors resonant frequency. In addition an analytical model is developed for the resolution of an oscillator detection system. The ultimate resolution of this type of system is derived and it is shown that there is an optimum Q for a given sensor design.

Chapter 4: Details the design and the measured results of a resonant strain gauge oscillator. This includes the development of a new time variant oscillator used to overcome issues of low Q sensor design.

Chapter 5: Reviews methods of frequency measurement. It discusses the trade-offs when using the different methods and proposes the use of a Sigma Delta phase locked loop (ΣΔPLL) to measure the output of the resonant strain gauge oscillator.

Chapter 6: Describes the design and measured results of a Sigma Delta PLL frequency measurement system used to measure the output of the resonant strain gauge oscillator developed in chapter 4.
2.1 MEMS DETF Resonant Sensors

2.1.1 Basic Operation

Resonant sensors have been used in numerous applications. For example they have been used to measure gas concentrations (mass) [6], acceleration [7], rotation [3], pressure [8] and strain [9]. In each of these applications the resonant frequency, $f_r$, is a function of the quantity to be measured. These quantities usually affect the resonant frequency by changing the stiffness (spring constant, $K_m$) or mass, $M_{eff}$ of the resonator. For instance a resonant accelerometer sensor (Fig. 2-1) operates by attaching a mass to one end of the double-ended tuning fork (DETF).

![Fig. 2-1: DETF accelerometer.](image-url)
2.2 DETF Sensor Properties

Acceleration applied to the mass in the axial direction creates a force, which results in an extension, or compression of the DETF beams. Hence changing the beams spring constant and thus their resonant frequency.

2.2 DETF Sensor Properties

This section will outline some basic properties of DETF sensors, which will be used to provide valuable insight into the design of these sensors in this and later chapters.

2.2.1 Sensitivity

As shown in Fig. 2-1, a force is applied to the DETF axially. If the force is a time varying signal, then the resonant frequency of the DETF is frequency modulated by the applied signal. The change in the resonant frequency due to a change in input is called the sensitivity. To find the sensitivity, the resonant frequency must be derived as a function of force applied to the beam.

![DEFT Sensor with Actuators](image)

*Fig. 2-2: DEFT sensor with actuators.*
Consider the DETF shown in Fig. 2-2. The Rayleigh method can be used to estimate the resonant frequency of the DETF beams. This method is based on equating the maximum kinetic energy to the maximum potential energy and solving for the resonant frequency [10].

\[
\omega_r^2 = \frac{K_{\text{eff}}}{M_{\text{eff}}} = \frac{EI \int_0^L \left( \frac{\partial^2 \phi(x)}{\partial x^2} \right)^2 \, dx + F_{1/2} \int_0^L \left( \frac{\partial \phi(x)}{\partial x} \right)^2 \, dx + A E \varepsilon_{\text{BI}} \int_0^L \left( \frac{\partial \phi(x)}{\partial x} \right)^2 \, dx}{\rho A \int_0^L \phi(x)^2 \, dx + M_{\text{act}}} \quad (2-1)
\]

Where \( F_{1/2} \) is the axial force applied to one DETF beam, \( E \) is the Youngs modulus and \( I \) is the beams moment of inertia. The density of the beam is \( \rho \), its length is \( L \), and its cross sectional area is \( A = w \times t \). Also, it is assumed that the beam will have some built-in strain, \( \varepsilon_{\text{BI}} \). \( M_{\text{act}} \) is the actuator mass. It is modeled as a lumped mass at the center of the beam. Finally, \( \phi(x) \) is a trial function, which estimates the mode shape of interest. If \( \phi(x) \) is the exact solution for the mode shape then this method gives the exact resonant frequency. Otherwise the Rayleigh method estimates an upper bound for the resonant frequency. For operation in the 1st mode the trial function (2-2) proposed by [11] was used in this work.

\[
\phi(x) = \frac{16}{L^4} x^2 (x - L)^2 \quad (2-2)
\]

Solving equation (2-1) for the resonant frequency we get:

\[
\omega_r^2 = \frac{256 E \varepsilon^3 t}{15 L^3} \left( \frac{1}{128} \rho t w L + M_{\text{act}} \right) \left( 1 + \frac{1}{7} \frac{L^2}{E w^3 t} F + \frac{2}{7} \frac{L^2}{w^3} \varepsilon_{\text{BI}} \right) = \frac{k_{\text{eff}}}{M_{\text{eff}}} (1 + \alpha F + \beta \varepsilon_{\text{BI}}) \quad (2-3)
\]
In equation (2-3) only a fraction of the beam mass, $\rho \cdot twL$, contributes to the total effective mass, $M_{\text{eff}} = \frac{128}{315} \rho \cdot twL + M_{\text{act}}$. Also, the effective spring constant $k_{\text{eff}} = \frac{256 \cdot E \cdot w^3t}{15L^3}$ agrees well with the spring constant of a clamped-clamped beam, $\frac{16E \cdot w^3t}{L^3}$ (assuming force is applied at the center of the beam). Finally $\omega_0 = \sqrt{\frac{k_{\text{eff}}}{M_{\text{eff}}}}$ is the nominal resonant frequency. Using equation (2-3), the force sensitivity can be calculated assuming a built in strain, $\varepsilon_{BI}$ in the DETF beams (for $F = 0$):

$$\frac{\partial \omega_r}{\partial F} = \frac{1}{2} \frac{\alpha \omega_o}{\sqrt{1 + \beta \varepsilon_{BI}}} = \frac{1}{14} \frac{256 \cdot L}{15 \cdot w^2} \left( \frac{1}{\frac{128 \cdot \rho \cdot twL + M_{\text{act}}}{315}} \right) \left( 1 + \frac{2 \cdot L^2}{7 \cdot w^2} \varepsilon_{BI} \right)$$

(2-4)

The sensitivity to strain, $\varepsilon$ can be calculated by noting that $F = 2AE\varepsilon$ and replacing force with strain in equation (2-3).

$$\frac{\partial \omega_r}{\partial \varepsilon} = \frac{1}{2} \frac{\beta \omega_o}{\sqrt{1 + \beta \varepsilon_{BI}}} = \frac{1}{7} \frac{256 E \cdot L}{15 w} \left( \frac{1}{\frac{128 \cdot \rho \cdot twL + M_{\text{act}}}{315}} \right) \left( 1 + \frac{2 \cdot L^2}{7 \cdot w^2} \varepsilon_{BI} \right)$$

(2-5)

It is interesting to note that the sensitivity of the DETF for both force and strain is inversely proportional to the actuator mass. In other words the highest sensitivity can be obtained by reducing the actuation mass to zero. In addition, increasing the length and or decreasing the width of the DETF beam improves sensitivity. This implies that lower frequency DETFs have better sensitivity.
2.2 DETF Sensor Properties

2.2.2 Temperature Sensitivity

Temperature sensitivity is an important consideration in the design of resonant sensors. For DETFs there are two main causes for temperature variation. The first is due to variation in the stiffness (Youngs modulus) with temperature. The second is due to mismatch in the coefficient of thermal expansion (CTE) between the sensor packaging and the DETF sensor.

The following relationship between Youngs modulus and temperature was published by [12] in 1982.

\[
E(T) = \left( 3.65 \times 10^{-6} \frac{\text{GPa}}{\text{K}} \right) T^2 - \left( 8.674 \times 10^{-3} \frac{\text{GPa}}{\text{K}} \right) T + 165.94 \text{GPa}
\]  

For: \( 293 \, ^\circ\text{K} \leq T \leq 923 \, ^\circ\text{K} \)  

(2-6)

A plot of (2-6) is shown in Fig. 2-3.

![Plot of Equation (2-6). Youngs modulus of polysilicon versus temperature.](image)

Fig. 2-3: Plot of Equation (2-6). Youngs modulus of polysilicon versus temperature.
Consider the range of temperature from 293 °K to 393 °K. In this range the modulus is a fairly linear function of temperature as shown in Fig. 2-3 and can be modeled by:

\[ E(T) = \gamma T + E_{0^\circ K} = -1.12 \times 10^{-2} \frac{\text{GPa}}{\text{oK}} T + 166.36 \text{GPa} \], \text{ For } 293 \text{ °K} \leq T \leq 393 \text{ °K} \quad (2-7)

The DETF temperature sensitivity is found by substituting (2-7) into (2-3) and taking the derivative with respect to temperature. Also it is assumed there is no built in strain.

\[
\frac{\partial \omega_r(T)}{\partial T} = \frac{\partial}{\partial T} \left( \frac{256 E(T) w^3 t}{15 L^3 (128 \rho t w L + M_{\text{act}})} \right) = \frac{1}{2} \frac{\partial E(T)}{\partial T} \omega_r(T), \text{ or}
\]

(2-8)

This relationship indicates that the percentage change in resonant frequency due to temperature is half that of the Young's modulus due to temperature. For example at 293 °K the change in resonant frequency due to temperature would be –34.3 ppm/°K versus a relative change in Young's modulus of –68.6 ppm/°K.

Mismatch in the CTE’s of the DETF and the packaging can also contribute to temperature variation. Consider the case where the DETF is to be used to sense strain in steel machinery. In this case the DETF is anchored to a silicon substrate and the substrate is bonded to steel. A strain is generated in the silicon substrate due to the difference in CTEs. Steel has a CTE of approximately 12 ppm/°K while silicon’s CTE is approximately 3 ppm/°K. Therefore the strain seen by the silicon substrate (2-9) will be tensile since steel expands four times more over temperature than the silicon.
2.2 DETF Sensor Properties

\[ \varepsilon(T) = (\text{CTE}_{\text{Steel}} - \text{CTE}_{\text{Si}}) \cdot T = \Delta \text{CTE} \cdot T \]  
(2-9)

Adding this temperature variation into (2-3) and taking the square root gives an equation for resonant frequency that captures both temperature effects:

\[ \omega_r = \sqrt{\frac{256}{15 L^3} \left( \frac{128}{315} \rho t w L + M_{\text{act}} \right) \left( 1 + \frac{2}{7} \frac{L^2}{w^2} (\varepsilon + \Delta \text{CTE} \cdot T) \right)} \]  
(2-10)

The normalized temperature sensitivity is given by:

\[ \frac{1}{\omega_r(T, \varepsilon)} \frac{\partial \omega_r(T, \varepsilon)}{\partial T} = \frac{2}{7} \frac{w^2}{L^2} \Delta \text{CTE} + \frac{\gamma}{2(\gamma T + E_{0^\circ K})} + \frac{2}{7} \frac{w^2}{L^2} (\varepsilon + \Delta \text{CTE} \cdot T) \]  
(2-11)

Equation (2-11) captures the total temperature variation of the resonant frequency. For example, a DETF strain sensor with a Length of 200 \( \mu \)m and a width of 6 \( \mu \)m attached to a steel substrate at 293 \(^\circ\)K, and assuming \( \varepsilon = 0 \), would have a temperature sensitivity of:

\[ \frac{1}{\omega_r(T, \varepsilon)} \frac{\partial \omega_r(T, \varepsilon)}{\partial T} = 1.16 \times 10^{-3} \text{ ppm/}^\circ\text{K} - 34.34 \text{ ppm/}^\circ\text{K} = -34.339 \text{ ppm/}^\circ\text{K} \]  
(2-12)

Thus the temperature variation due to the Youngs modulus dominates. However the variation due to the TCE mismatch cannot be ignored depending on the resolution requirements of the sensor.

2.2.3 Temperature Compensation

Since temperature variation can greatly affect the sensitivity and operation of a device there is a lot of interest in minimizing/removing it through various temperature
2.2 DETF Sensor Properties

compensation techniques. Variation due to CTE mismatch can be removed by clever package design. For instance, the sensor chip can be attached to the package with a compliant adhesive that absorbs all the strain due to package expansion. Another way to remove strain due to CTE mismatch would be to design the sensor such that transfer of strain from the chip substrate into the DETF is minimized. For instance, the tuning fork can be somewhat isolated from the strain by anchoring it at only one point. The other side of the tuning fork then can be connected through a compliant spring, $k_c$ to the substrate. The spring constant $k_c$ must be designed such that it is much less than the axial stiffness, $k_{axial}$ of the DETF. If this is done most of the strain will be absorbed by the spring $k_c$. For this type of compensation the strain transferred from the substrate to the DETF is:

$$
\varepsilon_{DETF} = \frac{k_c}{k_c + k_{axial}} \varepsilon_{Substrate} \tag{2-13}
$$

The only constraint on $k_c$ is to make sure the suspension meets the maximum shock (acceleration) requirements for the sensor.

Unfortunately it is not as easy to remove the temperature variation due to Youngs modulus for DETF resonant sensors. Consider the resonant accelerometer in Fig. 2-1. Assuming the temperature variation due to CTE mismatch has been removed by packaging:

$$
\frac{\partial \omega_x}{\partial a} = M_{accel} \frac{\partial \omega_z}{\partial F} = \frac{M_{accel}}{14} \sqrt{\frac{256}{15} \frac{L}{E t w^3} \frac{1}{1 + \frac{128}{315} \rho t w L + M_{act}}} \left(1 + \frac{2 L^2}{7 w^2} e_{BI}\right) \tag{2-14}
$$
Note, that the sensitivity is inversely proportional to the Youngs modulus. Hence it will vary with temperature. Also note that even using a differential design as proposed in [13] will not remove this variation. To compensate, the temperature of the device must be measured and sensor calibration must be used.

Unlike inertial sensors such as the resonant accelerometer outlined above, resonant strain sensors require the sensor packaging to transfer all the strain from the material it is attached to the DETF. As a result CTE mismatch cannot be reduced using isolation techniques. In addition, the strain sensitivity for DETF sensors (2-5) is proportional to the Youngs modulus and hence temperature. Therefore the strain sensor will be affect by CTE mismatch and variation in Youngs modulus due to temperature.

An example of a temperature compensation scheme for a resonant strain sensor is shown in Fig. 2-4. It is based on using two DETFs. One is temperature and strain sensitive. The second is denoted as the reference DETF is sensitive only to temperature. The reference DETF is decoupled from the substrate with a spring, $k_c$. Typically the two DETFs have different geometries i.e. beam width and length to prevent mode locking. In addition the orientation of the DETFs should be the same to avoid differences in Youngs modulus due to orientation. Two assumptions are made in this scheme. First $k_c \ll k_{\text{axial}}$ and hence no strain is applied to the reference DETF. This assumption is valid since $k_{\text{axial}}$ can be in the range of $10^4$ N/m to $10^5$ N/m. At the same time, $k_c$ can be in the...
range of $10^{-1}$ N/m to 10 N/m. If designed correctly the strain that reaches the DETF (2-13) can be up to a million times smaller than the strain applied to the sensor substrate. The second assumption is that the temperature of the reference DETF is the same as the strain sensitive DETF. This is a fairly good assumption as silicon has a high thermal conductivity. Also the reference DETF can be placed very close to the strain sensitive DETF. Applying these assumptions the resonant frequency for the reference DETF is:

$$\omega_{\text{ref}} = \frac{\sqrt{256 \left( \gamma T + E_{0^\circ K} \right) w^3 t}}{15L^3 \left( \frac{128}{315} \rho t w L + M_{\text{act}} \right)} = \sqrt{\frac{I_{\text{effi}} \left( \gamma T + E_{0^\circ K} \right)}{M_{\text{effi}}}}$$

(2-15)

The resonant frequency for the strain and temperature sensitive DETF is given by:
2.2 DETF Sensor Properties

\[ \omega_{\text{strain}} = \sqrt{\frac{I_{\text{eff}_1} \ (\gamma T + E_{0^\circ K})}{M_{\text{eff}_2}} (1 + \beta_2 (\varepsilon + \varepsilon_{\text{BI}} + \Delta \text{CTE} \cdot T))} \]  \hspace{1cm} (2-16)

Solving these equations for both temperature and strain, the solution for temperature can be shown to be:

\[ T = \frac{M_{\text{eff}_i} \omega_{\text{ref}}^2 - E_{0^\circ K} I_{\text{eff}_i}}{\gamma I_{\text{eff}_i}} = a_1 + b_1 \omega_{\text{ref}}^2 \]  \hspace{1cm} (2-17)

And the solution for strain is:

\[ \varepsilon = \varepsilon_{\text{BI}} + \frac{E_{0^\circ K} I_{\text{eff}_i} \Delta \text{CTE} - M_{\text{eff}_i} \Delta \text{CTE} \omega_{\text{ref}}^2}{\gamma I_{\text{eff}_i}} - \frac{1}{\beta_2} \left( 1 - \frac{I_{\text{eff}_i}}{I_{\text{eff}_1} M_{\text{eff}_1}} \left( \frac{\omega_{\text{strain}}}{\omega_{\text{ref}}} \right)^2 \right), \text{ or} \]

\[ \varepsilon = a_2 + b_2 \omega_{\text{ref}}^2 + c_2 \left( \frac{\omega_{\text{strain}}}{\omega_{\text{ref}}} \right)^2 \]  \hspace{1cm} (2-18)

Either a calibration scheme can be used to determine the coefficients of these equations, or the coefficients can be estimated using measured DETF parameters (length, width, etc).

Another way to perform temperature compensation for a strain sensor is to use two different size (width, length) DETFs that are sensitive to both temperature and strain. This is similar to the method that used a reference DETF. Now both ends of the reference are anchored to the substrate. The assumption is now that both DETFs are at the same temperature and undergo the same strain. Once again this assumption relies on how closely the DETFs can be placed together. For instance it requires the strain field to be constant over that distance. In this case, both the DETFs resonant frequencies can be
described with equation (2-16). The resulting equations (2-19) can be solved for temperature and strain:

\[
\omega_1 = \sqrt{\frac{I_{\text{eff}_1}(\gamma T + E_{0^\circ K})}{M_{\text{eff}_1}} \left(1 + \beta_1 (\varepsilon + \varepsilon_{\text{BI}} + \Delta TCE \cdot T)\right)}, \text{ and }
\]

\[
\omega_2 = \sqrt{\frac{I_{\text{eff}_2}(\gamma T + E_{0^\circ K})}{M_{\text{eff}_2}} \left(1 + \beta_2 (\varepsilon + \varepsilon_{\text{BI}} + \Delta TCE \cdot T)\right)}
\]

(2-19)

The solution is omitted due to its large size. Finally, the calibration scheme for this method is more complicated since there are more unknowns.

The resonant gyroscope is the only type of resonant sensor where the sensitivity is not a function of the Youngs modulus and hence temperature compensation may not be required [14].

There are other examples of temperature compensation in the literature. In [15] a mechanically compensated scheme is proposed. This is a good solution for resonators used as frequency references, however it is not immediately applicable to resonant sensors. This is due to the fact that the sensors mechanical structure must be integrated with the temperature compensation. In addition it is possible to use the spring tuning effect due to electrostatic actuation [16] to cancel out temperature variation in the sensors output. In this case the resonators output frequency is a function of the bias voltage applied, \(V_B\).
2.2 DETF Sensor Properties

\[ \omega_r = \sqrt{\frac{k_{\text{eff}} - k_{\text{el}}}{M_{\text{eff}}}} \approx \sqrt{\frac{k_{\text{eff}} - \frac{1}{2} \left( \frac{\partial^2 C(x = 0)}{\partial x^2} \right) V_B^2}{M_{\text{eff}}}} \]  \hspace{1cm} (2-20)

Where \( C(x) \) is the actuation capacitance. A feedback loop can then be used to stabilize the resonator frequency by measuring temperature and adjusting \( V_B \) to cancel out the resonators temperature variation. Another method would be to use a heater to adjust the resonator frequency [17]. In this case temperature, is measured and the power supplied to the heater is adjusted by a feedback loop to keep the resonant frequency constant over temperature.

2.2.4 Actuation and Sensing

Actuation and sensing is an integral part of resonant sensors because it provides a means for detecting changes in resonant frequency by monitoring changes in the DETF beam position (magnitude and phase) relative to a driving force. In addition the design of the actuation and sensing is used to select the desired resonance. Like most MEMS devices DETF, resonant sensors have multiple resonances (modes) with only one desired mode. For instance, to drive a DETF at its fundamental resonance (1st mode) the actuators should apply maximum force at the center of the beams. Similarly, to sense the 1st mode, the beam position sensing should be designed to maximize its sensitivity to deflection at the center of the DETF (Fig. 2-2). Generally, actuation and sensing should be designed to maximize excitation/detection of the desired mode while minimizing excitation/detection of all other modes [18]. The sensors in this work were designed to use the first mode of the DETF.
Capacitive actuation and sensing, was used in this work, as it is readily available in micromachining processes. The two common types of capacitive actuators/sensors available are based on parallel plate and the lateral comb drive [19] actuators. The determination of which type to use is based on many factors including MEMS process constraints, required force/voltage for actuation, linearity, and maximum deflection. For resonators linearity in the actuation and sense electrodes is an important factor. Nonlinearities in the actuation/sensing can cause instability in the DETF resonant frequency [16, 20] and distortion in its output [16, 21]. As a result comb drive actuators are preferred for their excellent linearity. However one of the disadvantages of using comb drive actuators is that they have a smaller dC/dx for a given actuator area/mass, and actuation voltage compared to parallel plate actuators. Hence they generate less electrostatic force. This becomes a problem when the stiffness or effective spring constant of the resonator increases. As the stiffness increases, the voltage required to actuate the resonator increases (assuming damping remains constant). Therefore high frequency resonators tend to use parallel plate actuation, to avoid the large voltages necessary to actuate them with comb drive actuators.

2.2.5 Electrical Model

A DETF resonator can be modeled as a linear mass-spring-damper system with capacitive actuation and sensing Fig. 2-5.
Fig. 2-5: DETF (top left), Linear mass-spring-damper model with capacitive actuation and sensing (top right), Electrical equivalent model for a DETF resonator (bottom).

Assuming linearity, a two-port model can be developed, and simple equations can be written to describe the structure. Consider the case where the sense voltage, $v_s = 0$. In this case, the position of the structure, $x$, and the output current, $i_s$, can be calculated. Beam position can be derived from the force balance equation:

\[
\frac{1}{2} \left( \frac{\partial (2C_d)}{\partial x} \right) (V_B - v_d)^2 = 2(k_{eff} x + b \ddot{x} + M_{eff} \dddot{x}), \text{ and}
\]

\[
x(s) = \frac{\partial C_d}{\partial x} \frac{V_B v_d(s)}{m_{eff} s^2 + b s + k_{eff}}
\]
The output current can be derived from the relationship between charge, $Q$ and current. Note that it is important to keep track of the mathematical sign for the currents [22].

$$i_s = \frac{\partial Q_s(t)}{\partial t} = -\frac{\partial (2C_s(t)V_B)}{\partial t} = -2\frac{\partial C_s}{\partial x} V_B \frac{\partial x(t)}{\partial t}, \text{ and}$$

$$i_s(s) = -\left\{2\frac{\partial C_s}{\partial x} \frac{\partial C_d}{\partial x} V_B^2\right\} s V_d(s)$$

$$\text{m}_{\text{eff}} s^2 + b s + k_{\text{eff}}$$

This can be repeated by setting the drive voltage, $v_d = 0$, to calculate the current $i_d$. By linear superposition the two responses can be added together to obtain a two-port model for the resonator (Fig. 2-6).

The two-port small signal electrical model for the linear system shown in Fig. 2-6 assumes the drive and sense capacitances are matched ($C_d = C_s$) [22]. The values for the components in the electrical model are:

$$R_x = \frac{b}{\eta^2} \quad (2-23)$$

$$L_x = \frac{M_{\text{eff}}}{\eta^2} \quad (2-24)$$

$$C_x = \frac{\eta^2}{k_{\text{eff}}} \quad (2-25)$$

$$C_o = C_d(x = 0) = C_s(x = 0) \quad (2-26)$$
Where $K_{\text{eff}}$ is the DETF spring constant and $M_{\text{eff}}$ is the effective mass, and $b$ is the damping coefficient of the DETF beam and actuator.

Unfortunately this model is an ideal one and does not account for interconnect and packaging related parasitics. To understand what effect parasitics have on the resonant sensor it is useful to examine the ideal electrical model with added parasitic capacitance and resistance.

![Two-port electrical equivalent DETF model](image)

*Fig. 2-6: Two-port electrical equivalent DETF model.*

Three parasitic components, $R_s$, $C_{ps}$, $C_{pd}$, and $C_{ft}$ have been added to the ideal model in Fig. 2-6. The resistance added in series with the resonator, $R_s$ will reduce the $Q$ of the resonator. The parasitic capacitors $C_{ps}$ and $C_{pd}$ adversely affect the noise performance of the resonant sensor [23]. Finally the feed-through capacitance, $C_{ft}$, if large enough can make measurement of the resonator output impossible [2]. In subsequent chapters the effects of these components will be discussed in greater detail.
2.2 DETF Sensor Properties

2.2.6 Electrostatic pull-in

Electrostatic pull-in is an important aspect in MEMS resonant sensor design as it limits the maximum bias voltage, $V_B$. Therefore, it places a lower limit on the resonator motional resistance (2-23). Calculation of the voltage at which pull-in occurs, $V_B = V_{PI}$, can often be complicated and require computer simulation. However, a simple analytical equation can be derived to estimate $V_{PI}$ for the DETF resonators used in this work. Consider Fig. 2-7 in which part of the DETF with comb drive actuation is shown.

![Fig. 2-7: Pull-in condition for a DETF driven with a comb drive actuator.](image)

In this case, the pull-in condition is shown in Fig. 2-7. This assumes the actuator is most compliant ($k_x < k_y$) in the $x$ direction and any movement of the actuator in the $x$ direction results in the largest nonlinear change in the electrostatic force, $F_x$. In contrast, a
change in actuator position in the y direction causes a change in the force $F_y$ (due to $C_5$ and $C_6$). If $L_{\text{gap}}$ is made large enough, $F_y \ll F_x$, and pull-in in this direction can be ignored. Writing the force balance equation for the x direction:

$$\frac{1}{2} N_f \left( \frac{\partial}{\partial x} \left( C_2 + C_4 - C_1 + C_3 \right) \right) V_B^2 = k_x x = \frac{k_0}{L_{ab}^2} x$$  \hspace{1cm} (2-28)

Where $N_f$ is the number comb fingers. The capacitances and $k_0$ are:

$$C_2 = C_4 = \frac{\varepsilon_o t L_{\text{OVL}}}{d_0 - x}, \quad \text{and} \quad C_1 = C_3 = \frac{\varepsilon_o t L_{\text{OVL}}}{d_0 + x}$$  \hspace{1cm} (2-29)

$$k_0 = \frac{4 E w^3 t}{3 L}, \quad [24]$$  \hspace{1cm} (2-30)

Solving (2-28) for the pull-in value for $x$ and $V_B$:

$$x = x_{\text{PI}} = \left( \frac{2}{\sqrt{3}} - 1 \right) d_0 \approx 0.393 d_0$$  \hspace{1cm} (2-31)

$$V_B = V_{\text{PIX}} \approx 0.49 \sqrt{\frac{d_0^3 k_0}{L_{ab}^2 2 N_f \varepsilon_o t L_{\text{OVL}}}} \approx 0.57 \sqrt{\frac{d_0^3}{2 N_f \varepsilon_o t L_{\text{OVL}}} \frac{E w^3 t}{L L_{ab}^2}}$$  \hspace{1cm} (2-32)

Most of the design variables in equation (2-32) are determined by the desired DETF frequency, and actuator size/desired motional impedance. As a result, minimizing $L_{ab}$ is an attractive way to maximize $V_{\text{PIX}}$. However, this can be difficult since it depends on other design parameters such as distance from the beam to the inside of the stationary comb, $L_{bc}$ and the comb finger length, $L_f$. Depending on the type of process, $L_{bc}$ can be a function of the minimum spacing and substrate contact size. In addition, the desired comb drive linearity sets constraints on the comb finger length ($L_f$), the overlap ($L_{\text{OVL}}$) and the
spacing from the tip of the comb finger to the stationary comb ($L_{\text{gap}}$). In Fig. 2-7 $L_{\text{ab}} = L_{\text{gap}} + L_f + L_{bc}$. Therefore minimization of $L_{\text{ab}}$ is a trade off between maximizing $V_{\text{pix}}$ and the desired actuator linearity.

### 2.2.7 Device Nonlinearities

There are two main physical effects caused by device nonlinearity. The first is distortion in the resonator output current due to nonlinear actuator capacitance. A complete analysis of this is given in [16, 21]. The second is the mechanical and actuation nonlinearities that result in a driving force, $f_d$, and a spring constant that is a function of displacement. These nonlinearities can affect the long-term stability of the resonant sensor output frequency. In addition they can limit the power that can be applied to the resonator.

Consider the force balance equation (2-33).

$$\frac{1}{2} \left( \frac{\partial^2 C(x)}{\partial x^2} \right) (V_B - V_d) = k_{\text{eff}}(x) x + b \ddot{x} + M_{\text{eff}} \dot{x}$$

(2-33)

The derivative of $C(x)$ can be replaced with a series expansion around $x = 0$ and, $k_{\text{eff}}$ with a nonlinear spring. Note that this force balance equation accounts for only one electrode (the drive electrode). An additional term would have to be added on the right side of (2-33) to include a sense electrode and any nonlinearity it introduces.

$$\frac{1}{2} \left( \frac{\partial^2 C(0)}{\partial x^2} + \frac{\partial^3 C(0)}{\partial x^3} x + \frac{1}{2} \frac{\partial^4 C(0)}{\partial x^4} x^2 + \frac{1}{6} \frac{\partial^5 C(0)}{\partial x^5} x^3 \ldots \right) (V_B - V_d) = k_{m_1} x + k_{m_2} x^2 + b \ddot{x} + M_{\text{eff}} \dot{x}$$

(2-34)

The terms that are even powers of $x$, on the left hand side of equation (2-34), generate static (DC) force, and force in phase with the drive voltage, $v_d$ at the fundamental
2.2 DETF Sensor Properties

frequency, $\omega_r$. The odd powers generate force that is in phase with position, $x$ and are therefore responsible for the electrostatic spring tuning effect. A describing function analysis can be performed in order to linearize equation (2-34). This is done by substituting $x$ and $v_d$ with: $x = |x_o| \sin(\omega_r t)$ and $v_d = |v_o| \cos(\omega_r t)$. Then all terms except for the terms at the fundamental are discarded resulting in a linear differential equation (2-35).

$$F = \frac{\partial C(0)}{\partial x} v_B v_d = (k_{m1} - k_{el}) x + \frac{3}{4} \left( k_{m3} - \frac{4}{3} k_{e3} \right) |x_o|^2 x + b \ddot{x} + M_{eff} \ddot{x}$$

(2-35)

From (2-35) a linearized spring constant, $k_{TOT}$, can be defined that is a function of the peak deflection of the resonator, $x_o$.

$$k_{TOT} = (k_{m1} - k_{el}) + \frac{3}{4} \left( k_{m3} - \frac{4}{3} k_{e3} \right) |x_o|^2$$

(2-36)

The resonant frequency can now be written as:

$$\omega_r = \sqrt{\frac{k_{TOT}}{M_{eff}}} = \sqrt{\frac{k_{m1}}{M_{eff}} \left( 1 - \frac{k_{el}}{k_{m1}} + \frac{3}{4} \left( \frac{k_{m3}}{k_{m1}} - \frac{4}{3} \frac{k_{e3}}{k_{m1}} \right) |x_o|^2 \right)}$$

(2-37)

The electrostatic spring constants $K_{e1}$ and $K_{e3}$ are:

$$k_{el} = \left( \frac{V_B^2}{2} + \frac{V_a^2}{8} \right) \frac{\partial^2 C(0)}{\partial x^2}$$

(2-38)

$$k_{e3} = \left( \frac{V_B^2}{16} + \frac{V_a^2}{96} \right) \frac{\partial^4 C(0)}{\partial x^4}$$

(2-39)

The nonlinear mechanical spring constant for a DETF resonator can be calculated by using the method outlined in [25]. Assuming the mode shape described by (2-2) the mechanical spring constant is written as follows:
These nonlinearities give rise to several interesting effects that have a direct impact on resonator performance. For instance, (2-37) quantifies how electrostatic spring tuning due to nonlinear actuators, and mechanical nonlinearity affect the resonant frequency. In chapter 3 this result will be used to develop models which predict how flicker noise can affect the long-term stability of DETF resonators. It will be shown how the dependence of $\omega_r$ on $V_B$ and $v_o$, and hence resonator deflection has an adverse affect on resonator phase noise and hence sensor resolution.

2.3 Discussion

This chapter has presented the basic concepts needed to design DETF resonant sensors. This material will be used extensively in later chapters. We will see that parasitic capacitance plays a large role in determining the overall resolution for a resonant sensor. In addition, it will be shown that feed-through capacitance may also affect resolution by limiting the minimum $Q$ one can use for a resonant sensor. Therefore when designing resonant sensors these capacitances should be minimized. Finally, we will see (in chapter 3) that resonant sensor nonlinearity can create noise that can adversely affect the long-term stability of resonant sensor oscillators.
CHAPTER 3:

RESONANT SENSOR MEASUREMENT SYSTEMS

3.1 Resonant Sensor Measurement Systems

This chapter discusses the different types of resonant sensing methods used to measure the resonant frequency of the sensor. Also, an analytical model will be developed to calculate sensor resolution for this measurement system.

3.2 Resonant Sensing Methods

There are three methods for detecting a change in the sensor resonant frequency, \( f_r \). The first is a slope detection method [1]. Slope detection is based on detecting changes in the magnitude of the sensors output, given a fixed input frequency, \( f_d \). A block diagram of a slope detection system is shown in Fig. 3-1a. Fig. 3-1b shows the change in output voltage, \( \Delta V_{out} \), due to a shift in resonant frequency. This detection system has a limited measurement bandwidth of \( \sim f_r/2Q \), which is due to the fact that the transient response of the resonant sensor does not change instantaneously. In fact, when the sensor resonant frequency changes, the amplitude exponentially settles to its new steady state value with a time constant \( \tau = 2Q/\omega_r \) [1, 26].
3.2 Resonant Sensing Methods

Fig. 3-1: Slope detection method.

In many cases this is a severe limitation. For example, consider a resonator that has a resonant frequency of 1MHz and a Q of 10000. The maximum bandwidth would be ~ 314 Hz for this case. Theoretically, if the resonant sensor frequency could be measured directly the maximum bandwidth of the sensor would be close to its resonant frequency (1MHz).

Then next method used to detect resonant frequency changes is phase detection (Fig. 3-2a). In this case, a fixed frequency, $f_d = f_r$, is used as an input. The resulting phase at the output of the resonator is compared to the input signals phase. The advantage of this method is that the phase of the sensor changes at the same rate as its resonant frequency. However, the frequency range over which phase changes linearly with frequency is approximately $f_r/2Q$ (Fig. 3-2b). Therefore, the available bandwidth $\sim f_r/2Q$. As a result, this method cannot take advantage of the rapidly changing phase.
The last method uses an oscillator (feedback loop) to force the sensor to resonate at its resonant frequency (Fig. 3-3a). The oscillator output frequency, $f_{osc}$, tracks the resonant frequency of the sensor by forcing the phase shift through the loop to be zero or $\Phi_{\text{LOOP}} = \Phi_{\text{DETF}} + \Phi_{\text{osc}} = 0$ (Fig. 3-3b). Ideally, $\Phi_{\text{osc}}$ is zero and hence the feedback loop tracks changes in the resonator phase. As a result, this method’s bandwidth is only limited by the bandwidth of the oscillator electronics. Because of the improvement in sensor bandwidth, this is the method used in resonant sensor measurement system developed in this research.
3.3 Resonant Frequency Measurement Using an Oscillator (Resolution)

Next, we explore the limits of resolution for an oscillator based frequency measurement system shown in Fig. 3-4. Resolution is one of the most important design criteria for evaluating resonant sensing. Therefore it is important to be able to estimate it analytically. For a resonant sensor that is placed in an oscillator, the output frequency of the oscillator is corrupted by noise from the resonator and electronics in the feedback loop. This noise determines the minimum detectable change in frequency (frequency resolution) and hence the minimum detectable strain, force or mass (sensor resolution). Therefore it is invaluable to have a good understanding of how the properties of both the resonant sensor and electronics in the oscillator feedback loop affect frequency noise.

![Oscillator detection method.](image-url)
3.3 Resonant Frequency Measurement Using an Oscillator (Resolution)

![Diagram of a resonant sensor with driving and sensing elements.](image)

Fig. 3-4: Oscillator based measurement system.

One approach used to determine an oscillators frequency noise is to measure or calculate its phase noise. The frequency noise can then be obtained from phase noise. This method has been used in the past by [1, 5] to estimate the resolution of resonant sensors. In [1, 5] phase noise is reported to be inversely proportional to both Q and drive level of the resonator, $x_b$. In [1], the minimum detectable frequency in hertz for an oscillator measurement system was reported to be:

$$\Delta f = \frac{1}{2\pi} \sqrt{\frac{2\pi f_r}{k_m Q x_b^2}} \frac{kT \cdot \text{BW}}{}$$

Where $\text{BW}$ is desired bandwidth of the sensor, and $x_b$ corresponds to the peak deflection, or drive level, of the resonator at the oscillators fundamental frequency of oscillation (or the sensor resonant frequency, $f_r$). In addition, $k_m$ is the resonators
effective stiffness, \(k\) is Boltzmann’s constant and \(T\) is temperature in degrees Kelvin. The frequency resolution in (3-1) is derived from a model for phase noise, which only accounts for \(1/f^2\) phase noise. Therefore it is only accurate over a limited sensor bandwidth where other components of phase noise are insignificant compared to \(1/f^2\). Assuming no \(1/f^3\) noise exists the bandwidth over which (3-1) is valid is approximately \(f_r/2Q\). In general, one must account for all components of phase noise to obtain a good estimate for resolution. Therefore, white phase noise, which determines the phase noise floor, and \(1/f^3\) phase noise must be included. When these components of noise are added to the model it can be shown that frequency resolution does not improve by arbitrarily increasing \(Q\), and \(x_b\), as indicated in (3-1), but in fact, there are values of \(Q\) and \(x_b\) at which resolution can be optimized.

This section develops a model for phase noise, which includes white phase noise, and \(1/f^3\) phase noise. In doing so, it reveals that there are values of \(Q\) and drive level that optimize sensor resolution, and these values are a function of the desired sensor bandwidth.

### 3.3.1 Phase Noise, Frequency Noise, and Sensor Resolution

For clarity, it is useful to explain the relationship between phase noise, frequency noise and sensor resolution. Phase noise is a measure of the noise in the side bands around an oscillators center frequency. Fig. 3-5a shows the power spectral density (PSD), \(S_x(f)\), of an oscillators output normalized by the power, \((P_o = S_x(f_r))\), contained in a 1 Hz bandwidth at the fundamental frequency of oscillation, \(f_r\). It has been shown that this
normalized spectrum is proportional to phase noise [27, 28]. If the oscillator spectrum is symmetric, the corresponding phase noise, \(S_\Phi\), is the sum of the noise power from the two side bands, \(P_{sb}\) (Fig. 3-5b). However, only half the noise power in the side bands contribute to phase noise [27, 28, 29] while the other half results in amplitude noise.

\[
\frac{S_\Phi(f_m)}{P_o} = \frac{2 L(f_m)}{f_r - f_m}
\]

\[
\frac{P_{sb}}{P_o} = 2 L(f_m)
\]

Note that \(L(f_m)\) is the single side band phase noise and is often used in specifications for communication systems. Since phase is the integral of frequency, the PSD for frequency, \(S_f\) can be determined from \(S_\Phi\).
3.3 Resonant Frequency Measurement Using an Oscillator (Resolution)

\[ S_f(f_m) = f_m^2 S_\Phi(f_m) \]  \hspace{1cm} (3-3)

To determine the frequency resolution, (3-3) is integrated over the sensor bandwidth, BW (Fig. 3-5c). Finally the sensor resolution can be found by dividing the frequency resolution in a given bandwidth by the sensitivity of the sensor.

\[ \text{Resolution} = \frac{1}{\frac{\partial f_r}{\partial q}} \sqrt{\int_{BW} f_m^2 S_\Phi(f_m) df_m} \]  \hspace{1cm} (3-4)

where the sensitivity of the sensor, \( \frac{\partial f_r}{\partial q} \) is the change in resonant frequency (\( f_r \)) with respect to the quantity to be measured (\( q \)). Therefore, if the phase noise for an oscillator is known, the corresponding sensor resolution can be determined.

*Fig. 3-6: Trans-impedance oscillator with noise sources.*
3.3 Resonant Frequency Measurement Using an Oscillator (Resolution)

3.3.2 Phase Noise Model for a Trans-impedance Oscillator (Optimum Q)

To design a resonant sensor oscillator it is important to understand how its resolution is affected by noise sources from both the sensor and the feedback electronics. It is useful to consider the phase noise of a trans-impedance oscillator (Fig. 3-6), as it is one of the simplest to analyze. Most of the insight gained from such an analysis applies to more complicated oscillators.

Fig. 3-6 consists of a MEMS resonant sensor whose output current is converted to voltage by a trans-impedance amplifier (TIA), with a gain of $-R_f$. A variable gain amplifier (VGA) and automatic level control circuitry (not shown) provide any remaining gain required while controlling the voltage level at the oscillator output, $V_{out}$. For oscillation, the loop must have a gain equal to $R_x$, where $R_x$ is the motional resistance of the resonant sensor. Therefore the VGA gain, $A_{vga}$ will be $R_x/R_f$. To calculate the phase noise for this oscillator we must calculate the ratio in (3-2). The side band power, $P_{sb}$, is proportional to all noise sources in the oscillator (Fig. 3-6) referred to its output ($v_{nout}^2$) around the center frequency $f_r$. Therefore the phase noise of the oscillator is:

$$S_P(f_m) = \frac{P_{sb}}{P_o} = \frac{v_{nout}^2}{1/2 v_{osc}^2} = \frac{1}{P_{MEMS} R_x} v_{nout}^2 = \frac{Q^2}{P_{eff} R_{eff} v_{nout}^2} \propto Q^2 v_{nout}^2 \frac{x}{B^2}$$

(3-5)

Where $V_{osc}$ is the peak output voltage of the oscillator, $P_{MEMS}$ is the power delivered to the resonator by the oscillator and $R_x$ is the resonators motional resistance. In (3-5) we have replaced $P_{MEMS}$ and $R_x$ with $P_{eff}$ and $R_{eff}$, which are independent of $Q$ allowing us to isolate the effect of $Q$ on phase noise. $P_{eff}$ and $R_{eff}$ are defined as:
3.3 Resonant Frequency Measurement Using an Oscillator (Resolution)

\[ P_{\text{eff}} = Q P_{\text{MEMS}} = \omega_r k_{\text{ml}} x_b^2 \]  

(3-6)

\[ R_{\text{eff}} = Q R_x = \frac{k_{\text{ml}}}{\omega_r \left( 2 \frac{\partial C_s}{\partial x} \frac{\partial C_d}{\partial x} \frac{V_{dc}^2}{x_b} \right)} \]  

(3-7)

Note that the following parameters are defined as:

- \( k_{\text{ml}} \): The linear term of the DETF beam spring constant.
- \( C_s \): Capacitance of one sense electrode
- \( C_d \): Capacitance of one drive electrode
- \( Q \): Quality factor of the resonator
- \( x_b \): Peak deflection of the DETF beams
- \( \omega_r \): Resonant frequency of the DETF in radians/sec
- \( V_{dc} \): DC bias of the resonator

The most important result of this analysis is (3-5) in which we see that phase noise increases with \( Q \) and decreases with increasing peak deflection, \( x_b \). Hence for a fixed \( x_b = x_{\text{max}} \), phase noise improves with decreasing \( Q \) assuming that \( \frac{V_{\text{out}}^2}{n_{\text{out}}} \) in (3-5) is independent of \( Q \).

The output noise of the oscillator in Fig. 3-6, \( \frac{V_{\text{out}}^2}{n_{\text{out}}} \) can be derived by modeling the resonant sensor as a second order linear system.
For small deviations from the resonant frequency $f_r$, $s=j\omega$ can be replaced with $s = j2\pi(f_r-f_m)$ in (3-8). The resulting output noise for small offsets from the center frequency is [27]:

$$
\frac{\nu^2_{\text{nout}}}{\nu^2_{\text{nres}}} = \left( \frac{f_o}{2Qf_m} \right)^2 \frac{\nu^2_{\text{nres}}}{A_{\text{vga}}} + A_{\text{vga}}^2 \left( 1 + \left( \frac{f_o}{2Qf_m} \right) \right)^2 \frac{\nu^2_{\text{nout}}}{\nu^2_{\text{nout}}} 
$$

(3-9)

Where $\nu^2_{\text{nres}}$ is the noise due to the resonator and $\nu^2_{\text{n_T}}$ represents all other noise sources referred to the node $n_T$, shown in Fig. 3-6. Assuming $R_f \times A_{\text{vga}} = R_x$:

$$
\frac{\nu^2_{\text{n_T}}}{\nu^2_{\text{nres}}} = \frac{\nu^2_{\text{nres}}}{A_{\text{vga}}} + \frac{\nu^2_{\text{nout}}}{A_{\text{vga}}} 
$$

(3-10)

$$
\frac{\nu^2_{\text{nres}}}{\nu^2_{\text{n_T}}} = 4kT R_x = 4kT \frac{R_{\text{eff}}}{Q} 
$$

(3-11)

$$
F = 1 + \frac{R_x}{A_{\text{vga}} Z_{\text{in}}} = 1 + \frac{R_{\text{eff}}\omega R_{\text{in}}}{A_{\text{vga}} Q} 
$$

(3-12)

Phase noise reduces to:

$$
S_{\phi}(f_m) = \frac{A_{\text{vga}}^2}{P_{\text{eff}}} \left( \frac{4kT Q}{A_{\text{vga}}} + \frac{Q^2}{R_{\text{eff}} A_{\text{vga}}} + \frac{1}{R_{\text{eff}} Z_{\text{in}}} + \frac{1}{A_{\text{vga}}} \right) \left( \frac{f_r}{2f_m} \right)^2 = \phi_{\text{NF}} + \phi f^2 
$$

(3-13)

At offset frequencies, $f_m$, greater than the oscillator loop bandwidth, $(f_r/2Q)$, $S_{\phi}(f_m)$ is equal to the phase noise floor, $\phi_{\text{NF}}$, or the level of white phase noise.
3.3 Resonant Frequency Measurement Using an Oscillator (Resolution)

\[ \Phi_{\text{nf}} = \frac{A_{\text{vga}}^2}{P_{\text{eff}}} \left( \frac{4kTQ}{A_{\text{vga}}} + \frac{Q^2}{R_{\text{eff}}} \left( \sqrt{\frac{v_{\text{na}}^2}{P_{\text{eff}}} + \frac{2}{v_{\text{vga}}}} \right) \right) \]  

(3-14)

For offset frequencies less than the loop bandwidth, the noise shaped by the resonator dominates (noise terms multiplied by \((f_{\text{r}}/2Qf_{\text{m}})^2\)), resulting in the region of phase noise referred to as \(1/f^2\) or \(\Phi_{1/f^2}\) noise.

\[ \Phi_{1/f^2} = \left( \frac{1}{f_{\text{m}}^2} \right) \frac{A_{\text{vga}}^2}{P_{\text{eff}}} \left( \frac{4kT (1 + A_{\text{vga}})}{QA_{\text{vga}}} + \frac{1}{R_{\text{eff}}} \left( \sqrt{\frac{v_{\text{na}}^2}{P_{\text{eff}}} + \frac{2}{v_{\text{vga}}}} \right) \right) \left( \frac{f_{\text{r}}}{2f_{\text{m}}} \right)^2 \]  

(3-15)

We can now use (3-13) to examine how Q affects phase noise. There are several interesting cases.

**Case 1: Optimum Phase Noise**

The optimum phase noise occurs when \(A_{\text{vga}} = 1\) and \(C_{\text{in}} = 0\). However, when \(C_{\text{in}}\) is non zero, the minimum phase noise is obtained when \(A_{\text{vga}} = 1\), and

\[ R_{\text{x}} = \frac{R_{\text{eff}}}{Q} = \frac{1}{\omega_0 C_{\text{in}}} = Z_{\text{in}} \]  

(3-16)

In this case phase noise (3-13) simplifies to:

\[ S_{\Phi}(f_{\text{m}}) = \frac{1}{\omega_0 k_{\text{m}}^2 b} \left( \frac{4kT + \omega_0 C_{\text{in}}}{4v_{\text{na}}^2 + v_{\text{vga}}^2} \right) Q + \frac{1}{Q} \left( \frac{4kT + \omega_0 C_{\text{in}}}{4v_{\text{na}}^2 + v_{\text{vga}}^2} \right) \left( \frac{f_{\text{r}}}{2f_{\text{m}}} \right)^2 \]  

(3-17)

It is apparent from (3-17) that the phase noise floor increases with Q while the \(1/f^2\) component of phase noise is inversely proportional to Q. As a result, resolution (3-4), which is proportional to the area under the curve in Fig. 1c, has an optimum with respect
Equation (3-18) suggests that as bandwidth increases the optimum Q goes down. This makes sense intuitively because as bandwidth increases, the phase noise floor, which is proportional to Q, becomes the dominant term in the calculation of resolution. As a result, phase noise and resolution can be optimized by setting \( Q = Q_{\text{opt}} \), \( R_x = Z_{\text{in}} \), \( A_{\text{vga}} = 1 \), and \( x_b = x_{\text{max}} \).

**Case 2: Amplifier noise is insignificant**

It can also be shown that even when \( R_x \neq Z_{\text{in}} \), \( A_{\text{vga}} \neq 1 \), and \( x_b \neq x_{\text{max}} \) resolution is optimized when \( Q \approx Q_{\text{opt}} \). Consider the case when the noise from the MEMS resonator is much larger than the noise contributions due to the amplifiers. Typically this occurs when the resonator has large a motional impedance. In this case phase noise simplifies to:

\[
S_\phi(f_m) = \frac{1}{P_{\text{eff}}} \left\{ \frac{4kTQA_{\text{vga}}}{2f_m} \left( \frac{f_r}{2f_m} \right)^2 + \frac{4kTQ}{Q} \left[ \frac{1 + A_{\text{vga}}}{Q} \right] \right\}^2
\]  

(3-19)

And the optimum Q is:
3.3 Resonant Frequency Measurement Using an Oscillator (Resolution)

\[ Q_{\text{opt}} = \sqrt{\frac{1 + A_{\text{vga}}}{A_{\text{vga}}} \frac{f_r}{2\text{BW}}} \quad (3-20) \]

**Case 3: Resonator noise is insignificant**

Finally when the resonator noise is small compared (small \( R_x \)) to the amplifier noise the phase noise simplifies to (3-21). In this case there is no optimum.

\[ S_{\Phi}(f_m) = \frac{A_{\text{vga}}^2}{P_{\text{eff}}} \left( \frac{Q^2}{R_{\text{eff}}} \left( \sqrt{n_a^2 + n_{\text{vga}}^2} + \sqrt{n_a^2 + n_{\text{vga}}^2} \right) + \frac{1}{R_{\text{eff}}} \left( \sqrt{n_a^2 + n_{\text{vga}}^2} \right) \left( \frac{f_r}{2f_m} \right)^2 \right) \quad (3-21) \]

### 3.3.3 Effect of Peak Deflection on Phase Noise

So far we have shown how \( Q \) affects phase noise. Now we examine how deflection affects phase noise in a resonant sensor. Equation (3-5) indicates that phase noise improves with increasing \( x_b \). Therefore, in order to get a resolution estimate one must determine the maximum possible deflection, \( x_{\text{max}} \). This deflection can be limited by mechanical characteristics of the resonator such as maximum distance the resonator can physically move. It can also be limited by system requirements such as the maximum available voltage used to drive the resonator. In many cases \( x_{\text{max}} \) is limited by resonator nonlinearities [30, 31, 32, 33]. These nonlinearities can result in phase noise that actually increases with increasing resonator deflection. In the following section we will detail these effects.
3.3.4 Critical Deflection and Phase Noise

There have been a number of publications [30, 31, 32, 33] on how MEMS resonator power is limited by mechanical and or electrostatic nonlinearities that result in a 3rd order spring constant, $k_3$. Most refer to [34] who determined the critical deflection, $x_c$, beyond which a system represented by (3-22) displays hysteresis known as duffing.

$$m \ddot{x} + b \dot{x} + k_{ml}x + k_3x^3 = F(t)$$  \hspace{1cm} (3-22)

The critical deflection is:

$$x_c = \frac{32 k_{ml}}{9 \sqrt{3} |k_3| Q} = \frac{x_{eff}}{\sqrt{Q}}$$  \hspace{1cm} (3-23)

According to [30, 31, 32, 33] this deflection, $x_c$, represents the maximum amplitude of oscillation beyond which the oscillation may become chaotic. Therefore the 3rd order nonlinearity can limit the maximum power delivered to the resonator, and hence resolution. In this case, the maximum power is:

$$P_{MAX} = \frac{\omega_c}{Q} k_{ml} x_c^2 = \frac{\omega_c}{Q^2} k_{ml} x_{eff}^2 = \frac{P_{eff}}{Q}$$  \hspace{1cm} (3-24)

This is an important result because $Q$ limits the maximum power! At this point however there is little evidence in the literature that this is the actual limiting phenomenon. Most work referring to this limit, report results for open loop measurements. However, when placing the resonator in a feedback loop (oscillator), the criteria for instability versus deflection may change and $x_c$ may not be the true limit. There is some evidence supporting this hypothesis. For instance, [35] demonstrated an
oscillator that operates with resonator deflections greater than $x_c$. In [35], operation beyond $x_c$ caused the oscillator phase noise to exhibit high $1/f^3$ phase noise and an even higher order noise ($1/f^4$). Clearly more research needs to be done to understand the effect of nonlinearities on oscillator performance.

Assuming deflection is limited by the duffing effect, we can replace $x_b$ in (3-17) with $x_c$. The phase noise now becomes:

$$S_{\Phi}(f_m) = \frac{1}{\omega_r k_{\text{mil}}^2 \varepsilon_{\text{eff}}} \left(4kT+\omega_r C_{\text{in}}\left(\frac{\nu_{na}^2}{\nu_{na}+\nu_{n\text{g}a}}\right)\right)Q^2 + \left(8kT+\omega_r C_{\text{in}}\left(\frac{\nu_{na}^2}{\nu_{na}+\nu_{n\text{g}a}}\right)\right)n_{\text{cg}}^{1/2} \left(\frac{f_r}{2\pi f_m}\right)^2$$ (3-25)

This is similar to the result in [33]. Note that there is no optimum with respect to $Q$, and increasing $Q$ will cause sensor resolution to get worse!

### 3.3.5 Peak Deflection and $1/f^3$ Phase Noise

It was shown above that phase noise has two components, $\Phi_{N_f}$ and $\Phi_{1/f^2}$, resulting from white noise in the oscillator loop. This analysis ignored the effect of flicker noise. Flicker noise or $1/f$ noise also contributes to phase noise, resulting in $\Phi_{1/f^3}$ phase noise. $\Phi_{1/f^3}$ occurs when nonlinearities mix the flicker noise up to the frequency of oscillation, $f_r$. In electro-statically driven resonators, the flicker noise is mixed up to $f_r$ through the transduction nonlinearity relating electrostatic force, $F_e$, to resonator drive voltage, $v_d$ (i.e. $F_e \sim \frac{1}{2} \frac{\partial C}{\partial x} v_d^2$). This noise is then converted to $\Phi_{1/f^3}$ phase noise by two mechanisms. The first is simply up-converted flicker noise shaped by the oscillator feedback loop, yielding $1/f^3$ phase noise.
Where $K_f$ is the aggregate flicker noise coefficient representing all flicker noise referred to the output of the oscillator (Fig. 3-6). The largest component of $1/f^3$ noise is caused by the second mechanism due to mechanical and or electrostatic nonlinearities. Specifically nonlinearities that result in a 3rd order spring constant, $k_3$. The resulting nonlinear spring constant causes the natural frequency, $\omega_n$, of the resonator to be dependent on its peak deflection $x_b$ as shown in (3-27).

$$\omega_n = \omega_r \sqrt{1 + \frac{3}{4} \frac{k_3}{k_{ml}} |x_b|^2} \approx \omega_r + \frac{3}{8} \omega_r \frac{k_3}{k_{ml}} |x_b|^2$$  \hspace{1cm} (3-27)

It can be shown [Appendix A] that the up-converted flicker noise modulates $x_b$, resulting in:

$$\Phi_2^{1/f^3} = \left(\beta \frac{1}{1/f^3}\right) \frac{1}{f_m^3} = \frac{9}{16} \left(\frac{k_3}{k_{ml}}\right)^2 \frac{K_f}{V_{dc}^2} x_b \frac{1}{f_m^3}$$  \hspace{1cm} (3-28)

From (3-28) it is obvious that the optimum $Q$ is not affected by the addition of $1/f^3$ noise to the model. Also, $1/f^3$ phase noise degrades with increased resonator deflection. Typically, $1/f^3$ phase noise is dominant at very small offsets from the oscillator center frequency. Hence for low bandwidth sensor applications, such as inertial sensing, maximizing $x_b$ may not improve sensor resolution because it will maximize $1/f^3$ phase noise as well.
3.3.6 Summary of Effect of Bandwidth, Q, and Resonator Deflection on Resolution

To summarize the effect of BW, Q and $x_b$ on phase noise and resolution, we can look at the case where amplifier noise is insignificant as an example since we obtain similar conclusions for the other cases detailed in section 3.3.2. In that case the oscillator phase noise (Fig. 3-7a) is:

$$
S_n(f_m) \approx \frac{1}{\alpha_r k_m l_b^2} \left( 4kT Q \left( \frac{A_{vga}}{Q} + \frac{1}{2 f_m} \right)^2 + \frac{9}{16} \left( \frac{f_r}{f_m} \right)^2 + \frac{k_f}{V_{dc}^2} \frac{x_b}{f_m^3} \right) \tag{3-29}
$$

and hence:

$$
(resolution)^2 \propto \frac{1}{\left( \frac{\partial f}{\partial q} \right)^2} \int_{\text{BW}} f_m^2 \left( \frac{Q}{x_b^2} + \frac{1}{Q x_b^2} + x_b^4 \right) \tag{3-30}
$$

Fig. 3-7b graphically illustrates how resolution, which is proportional to the area under the curve, is affected by the choice of, Q and $x_b$ for a given sensor bandwidth. We can see that increasing Q and $x_b$ arbitrarily does not optimize phase noise and sensor resolution. Instead, there is an optimum value for Q that is dependent on the bandwidth of the application:

$$
Q = Q_{\text{opt}} \propto \frac{f_r}{\text{BW}} \tag{3-31}
$$

In addition, deflection ($x_b$) should not be arbitrarily increased for all sensor bandwidths. For low bandwidth applications (BW $\leq f_{c32}$) (see Fig. 3-7), maximizing the resonator
deflection is not optimum, because $1/f^3$ noise increases. However for large bandwidth applications ($BW \geq f_{c2n}$) the noise floor dominates and $x_b$ should be maximized.

3.3.7 Ultimate Resolution for a Resonant Sensor

It is interesting to compare the frequency resolution predicted by (3-1) to the resolution predicted by this section. First note that in [1] it was assumed that all sources of noise are zero except for the resonator noise, $A_{vga} = 1$, and the resonator is linear (i.e. $k_3 = 0$). If we make these assumptions then equation (3-9) reduces to:

*Fig. 3-7: How $Q$ and $x_b$ affect phase noise and resolution.*
3.3 Resonant Frequency Measurement Using an Oscillator (Resolution)

\[
\overline{v_{\text{out}}^2}(f_m) = \left( \frac{f_r}{2Qf_m} \right)^2 \overline{v_{\text{res}}^2}
\]  

(3-32)

The phase noise from equation (3-13) becomes:

\[
S_\phi(f_m) = \frac{1}{P_{\text{eff}}} \left( \frac{4kT}{Q} \right) \left( \frac{f_r}{2f_m} \right)^2 = \frac{1}{\omega f \cdot k m \cdot x_b} \left( \frac{4kT}{Q} \right) \left( \frac{f_r}{2f_m} \right)^2
\]  

(3-33)

Integrating over the desired sensor bandwidth we obtain the frequency resolution in hertz:

\[
\Delta f = \frac{1}{2\pi} \sqrt{\frac{\omega f}{k m Q \cdot x_b} \cdot k T \cdot BW}
\]  

(3-34)

Thus we obtain the same result as [1] which is the minimum detectable change in frequency possible. Note that this prediction for resolution is unrealistic except for low bandwidth applications (BW << f_{c2n}) because of the assumptions made to derive it. Assuming the absence of a phase noise floor (i.e. no noise from electronics) or nonlinearity exists makes this prediction of resolution very optimistic! Further more, due to the above assumptions it does not predict that there is an optimum Q and x_b for a specified sensor BW!

A more realistic prediction would be to assume amplifier noise can be made insignificant but that the noise from the feedback resistor, R_f remains (Fig. 3-6). Also it can be assumed R_f = R_x (i.e. A_{vga} =1). We can also assume that the DC voltage source providing the resonator bias (Fig. 3-6) has a flicker noise source with a coefficient of K_f. In this case, phase noise reduces to the result shown in equation (3-29). Calculating the frequency resolution for this case we get:
\[ \Delta f \approx \frac{1}{\omega_1 k_{m1} x_b} \left( \frac{f_{\text{max}}^2 - f_{\text{min}}^2}{3} \right) + \frac{2kT\omega_0^2}{(2\pi^2 Q) f} \left( \frac{f_{\text{max}} - f_{\text{min}}}{f} \right) + \frac{9}{16} \left( \frac{k_3}{k_{m1}} \right)^2 \frac{k_f \ln f_{\text{max}}}{V_{\text{dc}}^2 x_b} \left( \frac{f_{\text{max}} - f_{\text{min}}}{f_{\text{min}}} \right) \] (3-35)

Where the bandwidth, BW, is equal to \( f_{\text{max}} - f_{\text{min}} \). This resolution model accounts for the existence of a phase noise floor and nonlinearity and therefore it predicts that resolution does not improve by arbitrarily increasing \( Q \) or \( x_b \).

### 3.4 Discussion

In this chapter we discussed the three different methods for detection of the sensor resonant frequency. It was shown that slope and phase detection are inferior to detection using an oscillator because of their limited bandwidth. In addition an analytical model was developed for the oscillator detection system. There are many important observations that can be inferred from this model. The most important is that there is a \( Q \) and deflection (\( x_b \)) that will optimize resolution of the sensor for a specified sensor bandwidth. The other ways to improve resolution follow directly from (3-13). They are:

1. Minimizing \( C_{\text{in}} \) and matching the impedance of the resonator with the input impedance \( Z_{\text{in}} \) (3-16).
2. Increasing the stiffness (\( k_{m1} \)) and/or resonant frequency of the resonator.
3. Reducing the ratio of \( k_3/k_{m1} \) (3-28) by reducing mechanical and electrostatic nonlinearities that contribute to \( k_3 \).
4.1 DETF Resonant Strain Sensor

The MEMS double-ended tuning fork (DETF) resonant strain sensor designed in this work is shown in Fig. 4-1. The sensor was fabricated with the Robert Bosch GmbH commercial surface micromachining process, which has two poly-silicon layers [36]. The first layer is used for routing and the second is the structural layer with 10.6 micrometer (µm) thick polysilicon. A photograph of one of the fabricated sensors is included in Fig. 4-2. Separate comb drives are used to excite the DETF into resonance and sense its motion. The measured DETF beam width is 5.67µm and its length is 200µm. Its measured resonant frequency, \( f_r \), was 217 kHz with a Q of 370 at atmospheric pressure [2]. To characterize the strain sensitivity of the resonators an electrostatic actuator that applies axial force and therefore produces an axial strain was added to the design (Fig. 4-1). The strain actuator is composed of 14 parallel plate actuators that are 496µm long and have a measured gap of 2.74 µm. Note that the actuator in Fig. 4-1 has only 10 parallel plate actuators. Table 1 summarizes the resonator parameters and its corresponding electrical equivalent model in Fig. 4-3.
The sensor was designed for a 10kHz bandwidth and hence, for this application the phase noise floor is the dominant source of noise affecting resolution. To minimize the phase noise floor, the sensor deflection, $x_b$, has to be maximized while operating at the optimum Q (see chapter 3). The DETF design and resonant frequency were kept close to previous work [5, 37] for comparison. Note that [5, 37] had resonant frequencies close to 200kHz. Therefore the optimum Q is approximately $200kHz/10kHz = 20$! Because the optimum Q for this design is less than the Q at atmospheric pressure it was decided to operate the sensor at atmospheric pressure. Hence avoiding the necessity of placing the sensor in a pressurized chamber.
Fig. 4-2: Photograph of DETF resonant strain sensor.

Table 4-1: DETF resonant strain sensor parameters.

<table>
<thead>
<tr>
<th>Process Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Device layer thickness</td>
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<tr>
<td>Minimum Gap (measured)</td>
<td>$g$</td>
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<table>
<thead>
<tr>
<th>DETF Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
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<tr>
<td>Number of Drive gaps</td>
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</tr>
<tr>
<td>Number of Sense gaps</td>
<td>$N_s$</td>
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</tr>
<tr>
<td>Comb finger overlap (drawn)</td>
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</tr>
<tr>
<td>Length (drawn)</td>
<td>$L$</td>
<td>200 µm</td>
</tr>
<tr>
<td>Width (measured)</td>
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</tr>
<tr>
<td>Parasitic Capacitance</td>
<td>$C_{ft}$</td>
<td>70 fF</td>
</tr>
<tr>
<td>Measured Q in air</td>
<td>$Q$</td>
<td>370</td>
</tr>
</tbody>
</table>
4.1 DETF Resonant Strain Sensor

<table>
<thead>
<tr>
<th>Calculated comb mass</th>
<th>$M_c$</th>
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<tbody>
<tr>
<td>Effective mass of DETF beam</td>
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<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; order spring constant</td>
<td>$k_{m1}$</td>
<td>636.7 N/m</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; order spring constant</td>
<td>$k_3$</td>
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</tr>
<tr>
<td>Calculated Resonant frequency</td>
<td>$f_{oc}$</td>
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</tr>
<tr>
<td>Measured Resonant frequency</td>
<td>$f_o$</td>
<td>217 kHz</td>
</tr>
<tr>
<td>Calculated Strain Sensitivity</td>
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</tr>
<tr>
<td>Measured Strain Sensitivity</td>
<td>$S_a$</td>
<td>39 Hz/µε</td>
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### Strain Actuator Parameters

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>Actuator length</td>
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<tr>
<td>Number of actuator gaps</td>
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<tr>
<td>Actuator $dC_a/dx$ (modeled)</td>
<td>$dC_a/dx$</td>
</tr>
</tbody>
</table>

---

Fig. 4-3: Electrical equivalent model of DETF resonant sensor ($V_{dc} = 84V$).
4.2 Challenges of low Q oscillator design (Feed-through Capacitance)

An oscillator feedback loop forces the DETF into resonance by sensing its output current and applying an appropriate force to maintain oscillation. One of the difficulties in making an oscillator with low Q MEMS devices is their high motional impedance, $R_x$ (Fig. 4-3). Due to this high impedance, the output of the MEMS device can be quite small. As a result, unwanted parasitic capacitances (such as $C_{ft}$) in parallel with the MEMS device (Fig. 4-3) can make measurement of the MEMS output impossible. Even with very small values of $C_{ft}$ the current through this capacitor can be many times larger than the DETF current. It can be shown that if the current through $C_{ft}$ exceeds two times the DETF current, oscillation with a sinusoidal-output oscillator is nearly impossible. The next section details how $C_{ft}$ causes sinusoidal-output oscillators to fail.

A sinusoidal-output oscillator has an amplitude control feedback loop that limits the peak output voltage, $V_{out}$, of the oscillator (Fig. 4-4). It does so by forcing the oscillator loop gain to be equal to one when the desired peak output level, $v_d$, is attained. Typically, a variable gain amplifier (VGA) is used in the oscillator feedback loop to adjust the loop gain. Therefore in steady state, when the oscillator loop gain is one, the VGA gain is:

$$A_{vga} = -\frac{R_x}{R_f}.$$  
In addition, the desired output level is set lower than the available supply voltage to avoid amplifier clipping. If designed correctly the output of the oscillator will be sinusoidal with a peak output voltage of $v_d$.

For oscillation to occur a sinusoidal oscillator consisting of a DETF and feedback electronics in Fig. 4-5a must meet two conditions. First, the loop gain at the frequency of oscillation should be greater than one. Second, the phase shift of the loop at that
frequency must be zero. In examining the effect of $C_{ft}$, the latter is most important. Typically, the phase shift through the electronic feedback circuitry is designed to be near zero degrees at the resonant frequency, $f_0$, of the DETF.

Thus allowing the rapid phase transition of $I_{sense}$ around the DETF resonant frequency (Fig. 4-5b) to determine the frequency of oscillation. However, if the parasitic feed-through capacitance, $C_{ft}$, is too large, the phase shift of $I_{sense}$ (Fig. 4-5b) will never reach zero degrees. As a result, oscillation may not be possible. Therefore the main issue for making an oscillator with low Q is overcoming the deleterious effects of parasitic feed through capacitance. It can be shown that the phase through the resonator will never reach zero degrees for:

$$C_{ft} > \max\left(C_{ft}\right) = \frac{Q}{2} C_x = \frac{1}{4\pi f_r R_x} \ \ \ \ \ \ (4-1)$$
Here $C_x$ and $R_x$ are the motional capacitance and resistance in the resonator linear model (Fig. 4-5a). As a result, we can determine the minimum $Q$ required for oscillation:

$$Q_{\text{min}} = 2 \frac{C_{ft}}{C_x} = 2 C_{ft} \frac{k_{ml}}{\left(\frac{\partial C}{\partial x} V_{dc}\right)^2}$$  \hspace{1cm} (4-2)$$

![Fig. 4-5: a) Simplified sinusoidal-output (linear) oscillator. b) Phase shift through the parallel combination of the resonator (RLC) and $C_{ft}$.]

For robust sinusoidal-output oscillator design we require $C_{ft} \approx \frac{\text{max}(C_{ft})}{5}$ or $Q \approx 5 \times Q_{\text{min}}$. For example the resonator presented in this work has a $C_{ft} = 70\,\text{fF}$ and requires a $Q$
in the range 6380 to guarantee oscillation (at a bias voltage of 84V). This is one of the
main reasons why resonant sensors need to be operated in a vacuum. However, by
increasing Q to ensure oscillation, resolution is adversely affected. For a sinusoidal-
output oscillator $C_{ft}$ can become the limiting factor with respect to phase noise and
resolution. This occurs when the minimum Q required for oscillation due to $C_{ft}$ is greater
than the value of Q needed for optimum resolution or:

$$Q_{opt} \geq 5 \times Q_{min} = 10 \frac{C_{ft}}{C_x}$$

(4-3)

For the strain sensor design in section 4.1, the feed-through capacitance would have to be
~ 0.2 femto-farads (fF) for $5 \times Q_{min} = Q_{opt}$. For our design at atmospheric pressure Q =
370. As a result, $5 \times Q_{min}$ must be less than or equal to 370 to use a sinusoidal oscillator.
This requires $C_{ft}$ to be less the 4fF. Therefore using a sinusoidal-output oscillator with
this sensor at atmospheric pressure will not work. We overcome this limitation by driving
the resonant sensor with a square wave instead of a sinusoid.

4.3 Square Wave Drive Oscillator

4.3.1 Operation

The operation of the SWO oscillator presented in [2] is based on the fact that when
the resonator and $C_{ft}$ are driven with a square wave, the components of $I_{sense}$ due to $C_{ft}$,
$I_{CB}$, and the resonator, $I_{DETF}$, can be separated in time (Fig. 4-6). This is accomplished by
detecting the first zero crossing of the $V_{sense} = R_s \times I_{sense}$, following a rising or falling edge
of the square wave (Fig. 4-6b).
4.3 Square Wave Drive Oscillator

Fig. 4-6: Illustration of how the parasitic component $I_{CT}$ can be separated by driving the sensor with a voltage step: a) Model to illustrate how separation of $I_{DET}$ and $I_{CT}$ is accomplished. b) Step response of model.

$I_{DET}$ determines the time of the zero crossing since $I_{CT}$ has decayed to zero at this point (Fig. 4-6b). Using this method [38], an SWO can enable oscillation in the presence of feed-through capacitances much larger than a sinusoidal oscillator will allow. Note that we define a sinusoidal oscillator as one in which the voltage used to drive the resonator is sinusoidal.

A square wave oscillator (SWO) circuit that accomplishes this separation is shown in Fig. 4-7a. The circuit applies a voltage step every half period (Fig. 4-7b). One half period after a rising or falling edge of $V_{Drive}$, the component due to $C_{ft}$ decays to zero. At this time, the contribution of the parasitic capacitance is zero and the DETF solely
determines the timing of the zero crossing of $V_{\text{sense}}$. A comparator determines when the zero crossing occurs, and applies a new voltage step to the resonator.

\[ \begin{align*} 
\text{Fig. 4-7: Example of SWO oscillator and its operation. a) SWO oscillator. b) Expected output waveforms of SWO oscillator.}
\end{align*} \]

The SWO oscillator used in this work (Fig. 4-8) uses an integrator front end. A second differentiator stage is used to obtain a zero degree phase shift at the resonant frequency. A comparator is then used to close the loop. An important design consideration is the maximum value of $C_\text{ft}$ that can be tolerated for proper operation. The maximum value of $C_\text{ft}$ that can be tolerated is:

\[ \max (C_\text{ft}) = \frac{1}{2} \frac{C_1 R_2}{R_3} \]  

(4-4)
Larger values of $C_f$ will cause the output of the second stage ($V_{Sense}$ in Fig. 4-8) to hit the maximum supply voltage. However, by changing the values of $C_1$, $R_2$ and $R_3$ the circuit can operate over a large range of $C_f$.

The magnitude (4-5) and phase (4-6) of the linearized circuit are, respectively:

$$ |H(s)| \approx G_c \left| \frac{R_1}{s R_1 C_1 + 1} \right| \left| \frac{s R_3 C_2}{s R_2 C_2 + 1} \right| $$  \hspace{1cm} (4-5)

$$ \arg(H(s)) \approx \frac{\pi}{2} - \text{atan}(\omega R_1 C_1) - \text{atan}(\omega R_2 C_2) $$  \hspace{1cm} (4-6)
Where $G_c$ is the comparator gain. To ensure oscillation, the component values for $H(s)$ must be chosen to such that the phase shift is close to zero at the sensor resonant frequency and the gain is greater than $R_x$. A plot of the magnitude and phase of $H(s)$ is provided in Fig. 4-9.

![Fig. 4-9: Magnitude and phase response of linearized SWO oscillator.](image)

Finally, this circuit will oscillate when the DETF bias is zero. When the bias is zero, the output current of the DETF is zero, and only $C_R$ determines the frequency of oscillation. In fact the circuit in Fig. 4-8 is an oscillator that is injection locked to the DETF's resonant frequency. However, in order for the circuit to lock correctly, the unlocked frequency, $f_{UL}$, must be designed such that it is greater than the natural frequency of the DETF. This ensures that $C_R$ does not affect the oscillation frequency when the resonator is biased. A lower bound for this frequency is given by:
Using these design equations the SWO can be designed for a variety of DETF resonant frequencies and feed through capacitances. A photograph of the PC board implementation of the SWO oscillator and its schematic are provided in Appendix B.

### 4.3.2 SWO Phase Noise

An analysis similar to the one described in chapter 3 for the trans-impedance oscillator can be done for the SWO oscillator. It is difficult to derive an analytical equation describing the output noise of the SWO oscillator (Fig. 4-8) due to the fact that its gain is both nonlinear and frequency dependent. Therefore we first quantify how the nonlinearity due to the comparator affects noise in the circuit.

Using the nonlinear SWO oscillator causes phase noise to be larger than it would be if a sinusoidal (linear) oscillator were used. This penalty due to the nonlinearity can be calculated using the simple models shown in Fig. 4-10. Consider the linear oscillator in Fig. 4-10a. The phase noise floor for this oscillator can be derived using equation (3-5). It is:

\[
\text{Linear Osc. Noise Floor} = \frac{v_{\text{out}}^2}{2v_{\text{osc}}^2} = \left( \frac{R_x}{R_g} \right)^2 \frac{v_{\text{noise}}^2}{2v_{\text{osc}}^2} = A^2 \frac{v_{\text{noise}}^2}{2v_{\text{osc}}^2}
\]

(4-8)

However for the nonlinear oscillator (Fig. 4-10b) the effect of the limiter must be taken into account.
When the limiter is operating in its linear region, it has a gain of $A$. This is true for voltages less than $V_s/A$ at the input of the limiter. However for greater voltages the gain of the limiter is zero. In steady state oscillation the input to the limiter will be equal to:

$$V_{\text{int}} = \frac{1}{A_L} \frac{4}{\pi} V_s \sin(2\pi f_r t) + v_{\text{noise}}$$  \hfill (4-9)

Assuming the noise term is much smaller than the sinusoidal term in (4-9), it can be shown that the noise, $\overline{v_{\text{noise}}^2}$, will be modulated by the effective gain shown in Fig. 4-11. As a result, if the noise bandwidth is greater than $f_r$, noise folding occurs, thus increasing the overall noise at the output of the oscillator.
4.3 Square Wave Drive Oscillator

Fig. 4-11: Simplified model for calculating affect of limiter on output noise of SWO oscillator: a) The time, $\tau$, over which $v_{\text{noise}}$ sees a gain of $A$. b) The time varying gain of the SWO limiter.

The phase noise floor of the nonlinear oscillator can be written as (assuming the noise is white):

$$\text{SWO Noise Floor} = \frac{2}{\sqrt{2}V_{\text{osc}}^2} \left( \frac{A_L}{2} \right)^2 \left( \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \sin\left( \frac{n A_L}{2} \right) \right)^2 = \frac{2}{\sqrt{2}V_{\text{osc}}^2} \frac{v_{\text{noise}}^2}{\frac{A_L}{2}^2} = \frac{2}{\sqrt{2}V_{\text{osc}}^2} \frac{v_{\text{noise}}^2}{\frac{A_L}{2}^2}$$  \hspace{1cm} (4-10)

Therefore the penalty for using the nonlinear oscillator is:

$$\frac{\text{SWO Noise Floor}}{\text{Linear Osc. Noise Floor}} = \frac{1}{4} \left( \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \sin\left( \frac{A L}{2} \right) \right)^2 = \frac{\gamma}{4}$$  \hspace{1cm} (4-11)

Using the values from Fig. 4-8 and Table 4-1, the noise penalty (4-11) was calculated to be 42. If the noise at the input of the limiter has a finite bandwidth that is smaller than the
frequency, $2 \times f_i$ (i.e. 1/f noise or filtered noise) the effect of folding on this noise is small. As a result for this noise $\gamma \approx 1$ and 1/f noise at the node, $V_{\text{int}}$, is multiplied by a gain of $(A_L/2)^3$ when referred to the output of the oscillator. Hence, the noise penalty for low frequency noise is:

$$\frac{\Phi_{1/f^3}^{(\text{SWO})}}{\Phi_{1/f^3}^{(\text{Linear})}} = \frac{1}{4}$$

(4-12)

This is an interesting result because it implies that 1/f noise is actually attenuated at the output of the nonlinear oscillator when compared to the 1/f noise at the output of the linear oscillator.

Using the results above we can estimate the phase noise of the SWO oscillator by simplifying the circuit in Fig. 4-8 to the linearized model in Fig. 4-10a. The nonlinearity due to the comparator folds high bandwidth ($\text{BW} >> f_i$) noise, multiplying it by a noise penalty. Noise that has an effective bandwidth much less than $f_i$ (i.e. 1/f noise or filtered noise) is not multiplied by the penalty. For instance noise due to the feedback resistor, $R_1$ and the resonator are low frequency noise. Thus they are not multiplied by the noise penalty. Typically these noise sources are added to the calculation, but in this case their noise contribution is insignificant compared to the other noise sources that are multiplied by the noise penalty. The significant noise sources due to the two amplifiers, $R_2$ and $R_3$ are referred to $V_{\text{int}}$ which approximates the total noise of the oscillator fairly well. Thus, the phase noise can be estimated with:
Where the output referred noise of the oscillator is:

\[ S_\phi (f_m) \approx \frac{1}{\sqrt{2}} \frac{1}{\sqrt{V_{osc}}} \left( \frac{\Delta L}{2} \right)^2 \left( \frac{V_{noise}^2}{V_{noise}^2} + \frac{\left( \frac{f_m}{2Q f_m} \right)^2}{2Q f_m} \right) + \frac{9}{16} \left( \frac{f_m}{k_m} \right)^2 \left( \frac{A_v}{2} \right)^2 \frac{K_f}{V_{dc}^2} \frac{4}{f_m^3} \]  \hspace{1cm} (4-13)

\[ V_{noise}^2 \approx \frac{V_{na}^2}{\sqrt{1 + \frac{R_s}{\omega_c} \left( \frac{C_{in}}{A_L} \right)^2}} + 4kTR_2 + \frac{4kTR_3}{G_{diff}^2} + \frac{V_{na}^2}{2} \]  \hspace{1cm} (4-14)

Note that \( G_{diff} \) is the differentiator gain and \( R_g \) is the gain of the integrator at the resonant frequency of the DETF. Also note that \( K_f \) is the flicker noise coefficient for the first amplifier in Fig. 4-8. Finally, \( V_{osc} = (4/\pi) \times V_s \). Where \( V_s \) is the peak value of the square wave driving the resonator. Using the values in Table 4-1, and Table 4-2 the phase noise can be estimated. A plot of the estimated phase noise is included in Fig. 4-13. Equation (4-13) predicts the single sided phase noise floor to be \(-120 \text{ dBc/Hz}\). Finally the SWO oscillator was simulated with a VGA replacing the comparator in Fig. 4-8 and setting \( C_{fn} = 0 \). Therefore if \( C_{fn} \) could be reduced enough to allow the use of a linear oscillator, even further reduction in phase noise could be achieved. The resulting linear oscillator had a simulated noise floor of \(-133 \text{ dBc/Hz}\), which is close to the expected value of \(-136 \text{ dBc/Hz}\) (Fig. 4-13).
*Table 4-2: Oscillator noise parameters.*

<table>
<thead>
<tr>
<th>Oscillator Noise Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peak Output Voltage</td>
<td>$V_s$</td>
<td>3.7 V</td>
</tr>
<tr>
<td>Differentiator Gain</td>
<td>$G_{\text{diff}}$</td>
<td>2.86 V/V</td>
</tr>
<tr>
<td>Integrator Gain</td>
<td>$R_g$</td>
<td>731 kΩ</td>
</tr>
<tr>
<td>$A_L$</td>
<td>$A_L$</td>
<td>25.5 V/V</td>
</tr>
<tr>
<td>Flicker Noise Coeff.</td>
<td>$K_f$</td>
<td>$4 \times 10^{-14}$ V²</td>
</tr>
<tr>
<td>Input Capacitance</td>
<td>$C_{\text{in}}$</td>
<td>3 pF</td>
</tr>
<tr>
<td>Parasitic Capacitance</td>
<td>$C_{\text{ft}}$</td>
<td>70 fF</td>
</tr>
<tr>
<td>Input noise OPA655</td>
<td>$v_{\text{na1}}$</td>
<td>6 nV/√Hz</td>
</tr>
<tr>
<td>Input noise AD8038</td>
<td>$v_{\text{na2}}$</td>
<td>8 nV/√Hz</td>
</tr>
<tr>
<td>Effective Gain = $G_c \times G_{\text{diff}}$ (Fig. 4-11)</td>
<td>$A$</td>
<td>2145 V/V</td>
</tr>
<tr>
<td>Time duration of gain (Fig. 4-11)</td>
<td>$\tau$</td>
<td>13.63 ns</td>
</tr>
<tr>
<td>Noise Penalty</td>
<td>$\gamma/4$</td>
<td>42.1</td>
</tr>
</tbody>
</table>

*Fig. 4-12: Calculated SWO phase noise.*
4.4 DETF Strain Sensor Measured Results

The SWO oscillator shown in Fig. 4-8 was implemented on a PC board (Appendix B). The measured oscillator output waveforms of both $V_{out}$, and $V_{sense}$, for resonator bias voltages of 28V and 84V are shown in Fig. 4-14. They agree well with results predicted in Fig. 4-7b. The SWO oscillator successfully resonates the DETF with a bias voltage as low as 28V despite a high $C_R$ (i.e. 70 fF) at atmospheric pressure (Q=370). At this bias the resonator has an effective $R_x \approx 162$ M$\Omega$. In this case an equivalent sinusoidal oscillator would require: 1) a Q larger than 57000 for $C_R = 70$ fF, or 2) a $C_R$ less than 0.5 fF if the Q is 370.

*Fig. 4-13: Simulated SWO phase noise.*
4.4 DETF Strain Sensor Measured Results

![Image of measured output waveforms of SWO oscillator.](image)

Fig. 4-14: Measured output waveforms of SWO oscillator.

To measure the strain sensitivity, an axial force was applied to the actuator as shown in Fig. 4-1. The resulting change in resonant frequency was measured with an Agilent 53131A frequency counter. The measured sensitivity, \( \frac{\partial f}{\partial \varepsilon} = 39 \text{ Hz/\mu} \varepsilon \), corresponds to the slope in Fig 4-15 and is in good agreement with the analytical model [9] that predicts a sensitivity of 36 Hz/\( \mu \varepsilon \) or 165 ppm/\( \mu \varepsilon \).
Next, a temperature-controlled chamber and a frequency counter were used to measure the temperature sensitivity of the oscillator (Fig. 4-16). The sensitivity was found to be $-6.8 \text{ Hz/ºC}$ or $-33 \text{ ppm/ºC}$. The expected temperature variation of the resonant frequency due to Young's modulus is approximately $-34 \text{ ppm}$ (chapter 2). Due to this large sensitivity, temperature compensation is necessary to achieve good resolution. In addition, this suggests that temperature can result in low frequency drift that may cause $1/f^3$ phase noise. An Agilent E5501A phase noise measurement system was used to measure the single sided phase noise, $L(f_m)$, of the oscillator with a bias voltage of 84V (Fig. 4-17).

**Fig 4-15:** Measured SWO frequency vs. strain (Sensitivity).
4.4 DETF Strain Sensor Measured Results

Note the dotted lines represent a curve fit for the measured data. The resolution can be calculated from the measured phase noise and is 18 nano-strain (nε) in a 10kHz bandwidth. If this calculation is done with only the phase noise floor we obtain the same result (18 nε). Thus for this bandwidth (10kHz), the \(1/f^2\) and \(1/f^3\) noise are insignificant and the phase noise floor dominates. Finally, the measured noise floor and \(1/f^2\) components match well with calculated and simulated results (Table 4-3 and Fig. 4-17).

The simulated \(1/f^3\) phase noise, due to amplifiers only, matches calculations. However it is insignificant compared to the measured \(1/f^3\) noise (Table 4-3). Therefore the measured \(1/f^3\) noise may be caused by some other \(1/f\) noise source other than the amplifiers. Alternately, it may be due to a nonlinear mechanism that has not been identified. At the

Fig. 4-16: Measured SWO frequency vs. Temperature.
moment the source of the large $1/f^3$ noise is not known but it is under investigation requiring a more sophisticated model of the oscillator.

*Table 4-3: Measured and calculated phase noise coefficients.*

<table>
<thead>
<tr>
<th>Phase noise Coefficients</th>
<th>Calculated</th>
<th>Measured (curve fit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_{NF}$</td>
<td>$1.66\times10^{-12}$ Hz$^{-1}$</td>
<td>$1.43\times10^{-12}$ Hz$^{-1}$</td>
</tr>
<tr>
<td>$\beta_{1/f^2}$</td>
<td>$2.45\times10^{-12}$ Hz</td>
<td>$2.39\times10^{-7}$ Hz</td>
</tr>
<tr>
<td>$\beta_{1/f^3}$</td>
<td>$1.77\times10^{-10}$ Hz$^2$</td>
<td>$4.02\times10^{-5}$ Hz$^2$</td>
</tr>
</tbody>
</table>

*Fig. 4-17: Measured and simulated phase noise of SWO oscillator.*
The measured noise floor for the SWO oscillator is –120 dBc/Hz. To the authors' knowledge this is the best noise floor published to date for a MEMS resonator in this frequency range. Table 4-4 provides a comparison of our results to recent work. Note that the oscillator has a better noise floor than previous works that include the use of integrated electronics (i.e. with low $C_{in}$ and low $C_{th}$). The results show clearly that high Q operation is not necessary to improve phase noise, as predicted in chapter 3.

*Table 4-4: Comparison of SWO oscillator phase noise floor to previous work.*

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrated Electronics, Vacuum</td>
<td>Both</td>
<td>Both</td>
<td>Both</td>
<td>Vacuum</td>
<td>None</td>
</tr>
<tr>
<td>Resonant freq.</td>
<td>175 kHz</td>
<td>16.5 kHz</td>
<td>145 kHz</td>
<td>57 kHz</td>
<td>217 kHz</td>
</tr>
<tr>
<td>Motional Resistance, $R_X$</td>
<td>Not Reported</td>
<td>3.8 MΩ</td>
<td>Not Reported</td>
<td>2.5 MΩ</td>
<td>18MΩ</td>
</tr>
<tr>
<td>Q</td>
<td>17000</td>
<td>51000</td>
<td>10000</td>
<td>14000</td>
<td>370</td>
</tr>
<tr>
<td>Phase noise floor (dBc/Hz)</td>
<td>-70</td>
<td>-70</td>
<td>-75(^1)</td>
<td>-83</td>
<td>-120</td>
</tr>
</tbody>
</table>

\(^1\)Taken from figure 9 in [5].

### 4.5 Discussion

This chapter demonstrated that operation at or close to the optimum Q derived in chapter 3 can improve phase noise and hence resolution of resonant sensors. The main
obstacle to operation at the optimum $Q$ is the parasitic feed-through capacitance ($C_{ft}$). The resonant sensor in this work would require a $C_{ft}$ less than $4\text{fF}$ for a sinusoidal-output oscillator to oscillate successfully with a $Q = 370$. Implementing a resonator with such a small feed-through capacitance is difficult even with integrated electronics and differential resonator topology. We solved the feed-through problem with the SWO oscillator. The comparator nonlinearity causes the SWO oscillator noise to be multiplied, thus incurring a noise penalty when compared to sinusoidal-output oscillators. However, even with this penalty the SWO oscillator performed better than previous work (Table 4-4).
CHAPTER 5:

FREQUENCY MEASUREMENT TECHNIQUES

5.1 Introduction

One of the goals of this research was to develop a DETF strain measurement system with a resolution of 0.1 microstrain ($\mu \varepsilon$) in a 10kHz bandwidth. The first half of this measurement system (DETF oscillator) was presented in chapter 4. However the DETF oscillator alone is not a complete measurement system (Fig. 5-1) because it does not provide a method for demodulating and digitizing the output of the oscillator.

Fig. 5-1: Oscillator based measurement system.
In this chapter a circuit block is presented that performs these functions. The specifications for the frequency measurement block are first defined, followed by an evaluation of several methods of frequency measurement. Finally, our method of choice for frequency measurement is presented.

### 5.2 Frequency measurement requirements: Strain Measurement System

In this section we define the frequency and/or period resolution required for the strain measurement system. The system is to have 0.1 με (rms) resolution over a 10kHz bandwidth. The DETF oscillator presented in chapter 4 had a resolution of 18 nε in 10kHz. Therefore the frequency demodulation/measurement block must contribute less than 98 nε to the overall resolution to meet the resolution criteria. In other words, the measurement system must have a frequency resolution, $f_{\text{res}}$, of less than 3.8 Hz or a period resolution, $t_{\text{res}}$, of less than 82 picoseconds (ps).

$$f_{\text{res}} = \frac{\partial f_r}{\partial \varepsilon} \varepsilon_{\text{res}} = 39 \frac{\text{Hz}}{\mu\varepsilon} \times 98 \text{n}\varepsilon \approx 3.8 \text{ Hz} \quad (5-1)$$

$$t_{\text{res}} = \frac{f_{\text{res}}}{f_r} = 39 \frac{\text{Hz}}{\mu\varepsilon} \times 98 \text{n}\varepsilon \times \frac{1}{(217 \text{ kHz})^2} \approx 82 \text{ ps} \quad (5-2)$$

For a 0.1με resolution in 10kHz, the signal to noise ratio (SNR) for the frequency measurement system needs to be greater than 77 dB for a full-scale range of ± 1000 με. Therefore, a ~13 bit frequency and/or time to digital converter is required for this application.
5.3 Frequency measurement methods

5.3.1 Digital Radio: Digitization Followed by Digital FM Demodulation

This method of operates by digitizing the entire spectrum of the signal. Then demodulation is performed digitally. This type of architecture (Fig. 5-2a) is now starting to be implemented in car radios [40]. By performing the demodulation digitally, unlike previous radio designs, these radios avoid the need to design separate paths for both AM and FM demodulation that previous radio designs had used. This type of flexibility is not necessary for resonant sensor since the output of the sensor is FM modulated only.

a)

![Diagram showing the basic architecture of AM/FM Digital Radio.]

b)

![Diagram showing the relationship between A/D converter SNR and phase noise.]

Fig. 5-2: AM/FM Digital Radio. a) basic architecture b) Relationship between A/D converter SNR and phase noise.
The requirement of digitizing the entire spectrum forces the converter to operate at sample frequencies that are several multiples of the carrier frequency of the FM signal. In addition, the FM spectrum is typically much larger than the bandwidth of the modulating signal (see Carson's Rule, [41]). In our sensors case the FM spectrum bandwidth is 98kHz while the sensor bandwidth is 10kHz. From a standpoint of sample rate (power) and bandwidth this method is not optimal. It must operate a sample rates that are much greater than the Nyquist rate \((f_N=20kHz)\) while obtaining no benefit from oversampling \((f_s > f_N)\).

In addition to the high sample rate the A/D converter must convert the FM spectrum of the oscillator without adding additional phase noise. In other words, the digitized oscillator spectrum SNR should not be affected by the A/D quantization noise. It can be shown that the SNR of the A/D converter is related to the phase noise floor the oscillator (Fig. 5-2b) by (5-1), where, \(f_s\), is the sample rate of the converter.

\[
\text{SNR} = \frac{1}{\Phi_{NF} f_s^2}
\]  

(5-1)

The measured phase noise floor of the DETF oscillator was \(\Phi_{NF} = -120\ \text{dBc/Hz}\) (Chapter 4). As a result, for the A/D converter quantization noise level equal to the oscillator noise floor, the converter SNR must be \(-65 \text{ dB} \) \((f_s = 3\times f_c = 651kHz)\). Therefore, an 11-bit A/D converter is required for our application. Also, to meet the specification of our design the digital demodulator must achieve an SNR of better than 77 dB. One of the best digital demodulators reported to date achieves a 74.7 dB SNR and a 73.8 dB signal to noise and distortion ratio (SNDR) [42]. As a result, this method requires a fairly complicated A/D
5.3 Frequency measurement methods

converter design, running at high sample rates, which is not optimal in terms of signal bandwidth or power. Finally, it requires a digital demodulator that achieves an SNR better than any reported to date.

5.3.2 Phase Locked Loop (PLL) demodulation

In the past, frequency demodulation for resonant sensors has been done with phase locked loops [43] (Fig. 5-3). In the system shown in Fig. 5-3, a voltage-controlled oscillator (VCO) is used to estimate the incoming frequency. The VCO is placed in the feedback loop of the PLL. Hence the output of the PLL, $V_{vco}$, is a function of the scale factor, $K_{vco}$ that relates the VCO’s output frequency ($f_{vco} = K_{vco} \times V_{vco}$) to its input voltage.

![Fig. 5-3: Oscillator based measurement system.](image)

The VCO voltage output, $V_{vco}$, is equal to the demodulated output of the sensor and can be digitized with an A/D converter as shown in Fig. 5-3. In order to digitize the output, $V_{vco}$, a 13-bit analog to digital converter is required to meet the 77 dB signal to noise ratio for our application. Like digital radio (above), a high resolution A/D converter must be designed when using this method. However unlike digital demodulation which required the A/D to operate at high sampling rate ($f_s \approx 3 \times f_i$) this method can operate the A/D at the Nyquist rate i.e. $f_s = 2 \times 10$ kHz.
5.3 Frequency measurement methods

One important aspect of this system is the quality of the VCO. The phase noise of the VCO appears at the output of the PLL [44] in addition to the phase noise of the DETF oscillator. Therefore its phase noise must be better than the resonant sensor oscillators to ensure that the overall resolution of the measurement system is not affected by the VCO. This can be particularly important for phase noise that is close to the carrier frequency. This noise is inversely proportional to Q. Hence the VCO would require a Q > 370 assuming it has a similar output noise and power as the DETF oscillator. Unfortunately, high Q (>100) oscillators are difficult to implement in standard CMOS processes. The second limitation with the VCO is that its scale factor, $K_{vco}$, is a function of temperature (T), power supply voltage ($V_s$) etc. Also, the VCO’s output frequency is usually a nonlinear function of its input voltage i.e. $K_{vco}$ is a function of $V_{vco}$. For the PLL shown in Fig. 5-3, $V_{vco}$ is equal to:

$$V_{vco} = \frac{f_{osc}}{K_{vco}(T, V_s, V_{vco})}$$

(5-2)

where $f_{osc}$ is the frequency of the DETF oscillator. Therefore, as indicated by (5-2), variation in $K_{vco}$ can degrade overall sensor performance.

This system improves on digital demodulation by demodulating the input first before digitizing it, thereby reducing the sample rate of the A/D converter. However it does not reduce the resolution required by the A/D and it introduces phase noise, nonlinearity, and variation due to the VCO.
5.3 Frequency measurement methods

5.3.3 Period measurement

Unlike the methods described thus far, period measurement techniques accomplish digitization and demodulation of a FM modulated signal by measuring/digitizing the time between zero its crossings. Hence period measurement combines demodulation and digitization in one step. Converters that perform this operation are generally referred to as time to digital converters (TDCs). The simplest method of period measurement is shown in Fig. 5-4a. In this example, digitization is accomplished by counting the number of reference clock periods that occur during the time between each zero crossing of the oscillator output. This method can digitize every ½ period of the oscillator output (Fig. 5-4b).

![Diagram of basic period measurement system]

**Fig. 5-4: Basic period measurement system.** a) block diagram. b) signal waveforms.
Therefore its sample rate is two times the resonant frequency of the oscillator, $f_r = 217$ kHz. Again, the sensor bandwidth, $BW$, needed is 10kHz. Hence the converter is oversampling by $f_r/BW$ or 21.7 times. This inherent oversampling available in period measurement systems can be used to reduce TDC complexity. In fact, by oversampling, the LSB ($1/f_{\text{ref}}$) of the TDC can be increased while still meeting the 82 ps (rms) time resolution, $t_{\text{res}}$, required by this application. Since the system in Fig. 5-4 makes a quantization error at the beginning and end of the measurement the total quantization noise made by the TDC is $2/\sqrt{12 f_{\text{ref}}}$. Using this result the clock frequency, $f_{\text{ref}}$, required to meet 82 ps resolution can be calculated (assuming the sample rate is $2 \times f_r$):

$$f_{\text{ref}} = \frac{2}{t_{\text{res}} \sqrt{12}} \sqrt{\frac{BW}{f_r}} = \frac{2 f_r^2}{(\partial f_r/\partial t_{\text{res}}) t_{\text{res}} \sqrt{12} f_r} \sqrt{\frac{BW}{f_r}} \approx 1.5 \text{GHz} \quad (5-3)$$

This means that the LSB of the converter must be ~668 ps. Even with oversampling this method requires a fairly high frequency clock. To avoid this, the Nutt method for time measurement can be used [45]. This method reduces the clock frequency needed by using a 3 step conversion of the time period to be measured, $T_m$. The first step of the conversion is similar the previous system shown in Fig. 5-4. It uses a counter to obtain a course measurement of $T_m$ with a quantization step of $1/f_{\text{ref}}$. The second and third steps measure the time $T_a$, and $T_b$ with a quantization step of $1/(N \times f_{\text{ref}})$. 
5.3 Frequency measurement methods

The total digitized time, $T_D$, measured by the Nutt TDC is (5-4) where $N_c$ is the output of the counter, and $N_a/N_b$ are the result of digitizing the time $T_a/T_b$ with an LSB of $1/(N \times f_{ref})$.

$$T_D = T_c + T_a - T_b = \frac{N_c}{f_{ref}} + \frac{N_a}{N f_{ref}} - \frac{N_b}{N f_{ref}} = T_m + \text{quantization error}$$  \hspace{1cm} (5-4)

The quantization steps for the measurement of $T_a$, and $T_b$ are obtained by dividing the clock with an N-tap delay line. Typically this delay line is implemented using a delay locked loop (DLL), which employs feedback to force the total delay through the delay line to be equal to one clock period. A block diagram of a Nutt converter is shown in Fig. 5-6. The required LSB for this method is still 668 ps. However by setting $N = 30$ the required clock frequency can be reduced to $\sim50$MHz.

*Fig. 5-5: Timing diagram of Nutt time measurement method.*
Time to digital converters can be used for the frequency measurement block of this sensor. They perform demodulation and digitization of the signal at the same time, which is an advantage over the previous methods. However, they must divide the reference clock (i.e. Nutt Method) to achieve high time measurement resolution while maintaining low clock frequencies. Thus the main challenge in TDC design is the design of these dividers (delay lines). The delay lines must have matched delays in the pico-second range. Therefore variation and mismatch in these delays is one of the main problems with TDC’s [46]. It is not unusual for a TDC to have values of INL and DNL that are multiples of 1 LSB and therefore most require calibration to remove the nonlinearities [46, 47]. In the following section we will see that high precision period measurement can
be accomplished without delay lines while using even lower clock frequencies. This is accomplished by shaping the quantization noise of the period measurement system.

5.3.4 Sigma Delta Phase Locked Loop

The third frequency measurement method uses a sigma delta phase locked loop (ΣΔPLL). A sigma delta phase locked loop (Fig. 5-7) is a hybrid of a PLL and the period measurement system detailed in Fig. 5-4 [48]. Similar to the period measurement system described in the previous section, the ΣΔPLL demodulates and digitizes the input in the same step.

![Fig. 5-7: Block diagram of a sigma delta PLL measurement system.](image)

It does so by controlling the counter period such that its output, $C_{out}$, is in phase with the input signal, $V_{out}$. The loop operates by measuring the difference in time ($e_n = t_n - \tau_n$) between the zero crossings of $V_{out}$ and $C_{out}$. Then it uses $e_n$ to estimate the length of time until the next zero crossing of $V_{out}$ (Fig. 5-8). Hence $N_{out} \times T_{ref}$ is an estimate of $T_{osc}/2$ where $T_{osc}$ is the period of $V_{out}$. 
5.3 Frequency measurement methods

Unlike a VCO, the counter cannot create a signal with an arbitrary period, but can only generate periods that are integer multiples of \( T_{\text{ref}} \). Therefore the \( \Sigma\Delta \) PLL makes a quantization error, \( q_n \), each time it estimates the sensor period. As a result, the output, \( N_{\text{out}} \), of the \( \Sigma\Delta \) PLL for a given sample is:

\[
N_{\text{out}} \approx \frac{T_{\text{osc}}(n)}{2T_{\text{ref}}} + q_n
\]  

(5-5)

The \( \Sigma\Delta \) PLL samples the period at every zero crossing, or twice the resonant frequency of the sensor. As a result the oversampling ratio, \( \text{OSR} = f_s/(2 \text{ BW}) \), for this converter is 21.7 or \( f_s = 434\text{kHz} \) and \( \text{BW} = 10\text{kHz} \).

The major difference between the period measurement system and the \( \Sigma\Delta \) PLL is that the \( \Sigma\Delta \) PLL loop shapes or moves the quantization noise power out of the band of interest.
As a result, noise shaping allows the use of a TDC with much larger quantization error (lower $f_{ref}$) while still meeting the resolution goal within the sensor bandwidth (Fig. 5-9).

![Diagram of noise shaping]

*Fig. 5-9: Example of noise shaping: a) unshaped quantization noise. b) quantization noise shaped or moved out of the signal band by the sigma delta loop.*

5.4 Discussion

This chapter reviewed some of the frequency measurement methods available to demodulate and digitize the output of a resonant sensor. Among the methods reviewed, the ΣΔPLL offers the simplest solution for our frequency measurement application. It will be shown in the following chapter that the ΣΔPLL can meet the 82 ps resolution with a clock frequency in the range of 10-20MHz. This is accomplished without the use of delay.
lines. Only a simple digital counter is required. Hence it removes the nonlinearity issues associated with matching delays in a typical TDC. In addition, the required A/D converter (Fig. 5-7) is only 4-5 bits! This is a significant reduction in A/D size compared to the 11 to 13 bits required for other methods. Because of these advantages, this is the method chosen to demodulate and digitize the sensors output.
6.1 Sigma Delta PLL Period Measurement System

This chapter will detail the design of the ΣΔPLL used to measure the output of the resonant sensor. It will describe how the linear model of the ΣΔPLL is developed. The required order of the sigma delta and the A/D converter size will be determined for our sensor application. Finally the design and measured results of a prototype ΣΔPLL implemented on a PC board will be discussed.

6.1.1 Linear Model of the Sigma Delta PLL

A good understanding of the ΣΔPLL’s basic operation is required to develop it’s linear model. Although its operation was described in chapter 5 part of it will be presented here to aid in the development of the linear model. One of the strengths of the ΣΔPLL (Fig. 6-1) is that it demodulates and digitizes the input frequency in the same step. It does so by controlling the counter period such that its output, \( C_{\text{out}} \), is in phase with the input signal, \( V_{\text{out}} \). The loop operates by measuring the difference in time (\( e_n = t_n - \tau_n \)) between the zero crossings of \( V_{\text{out}} \) and \( C_{\text{out}} \). The loop does this by using a phase detector to generate a pulse (Fig. 6-2) whose area, \( A_n \), is equal to \( K_d \times (t_n - \tau_n) \). Note \( K_d \) is the phase...
detector gain. \( A_n \) is then used to estimate the length of time, \( \Delta t_{n+1} = N_{n+1} \times T_{ref} \), until the next zero crossing of \( V_{out} \) (Fig. 6-2).

![Figure 6-1: Block diagram of a sigma delta PLL measurement system.](image)

As a result, there is a delay of one clock period between the time when the error, \( e_n = A_n/K_d \), is measured and when that measurement is used to estimate the next time period (Fig. 6-2). This results in a loop delay \((z^{-1})\) in the forward path of the loop (Fig. 6-3). To calculate the transfer function of the counter we note that the counter takes the input \( N_{out} \) and generates a time period equal to \( N_{out} \times T_{ref} \). The \( \Sigma\Delta \) PLL operates by forcing \( \tau_n \approx t_n \) where \( \tau_n \) and \( t_n \) are the zero crossings of the counter and DETF oscillator output respectively. Hence the transfer function of the counter must relate the counters input, \( N_{out} \), to its output \( \tau_n \). From Fig. 6-2 it is obvious that \( \tau_3 = \tau_2 + N_3 \times T_{ref} \), for \( N_{out} = N_3 \). Therefore the counter integrates period \( (N_{out} \times T_{ref}) \) to generate time, \( \tau_n \). As a result, in the z-domain transfer function of the counter is:

\[
\frac{\tau(z)}{N(z)} = \frac{T_{ref}}{1 - z^{-1}} \quad (6-1)
\]
The output of the linear model shown in Fig. 6-3 is:

\[ N_{out}(z) = \left[ \frac{G_L}{1 + G_L} \right] t(z) \left[ 1 - z^{-1} \right] T_{ref} + \left[ \frac{1}{1 + G_L} \right] q(z) \]  

(6-2)

where \( G_L \) is the loop gain \( (G_L = K_d A_Q z^{-1} F(z) C(z)) \), \( q \) is the quantization noise introduced by the counter, \( K_d \) is the phase detector gain and \( A_Q \) is the A/D converter (quantizer) gain.
It can be shown (Fig. 6-2) that \( t_n = t_{n-1} + \Delta t_n \). Where \( \Delta t_n \) is the time between zero crossings \( t_n \) and \( t_{n-1} \) and, \( \Delta t \) is equal to one half the sensors input period, \( T \). Thus \( t(z) \) can be related to the input period, \( T(z) \), as shown in (6-3). Substituting (6-3) into (6-2) results in the output, \( N(z) \) with respect to the input period (6-4).

\[
T(z) = 2 \times t(z) \left(1 - z^{-1}\right) 
\]

(6-3)

\[
N(z) = \left[ \frac{G_L}{1 + G_L} \right] T(z) + \left[ \frac{1}{1 + G_L} \right] q(z) 
\]

(6-4)

With sufficiently high loop gain in the signal band, the output of the \( \Sigma \Delta PLL \), \( N(z) \), is equal to the input period of \( V_{out} \) divided by \( 2 \times T_{ref} \). In addition the quantization noise in the signal band is attenuated by the loop gain.
6.1 Sigma Delta PLL Period Measurement System

6.1.2 High-Order Loop Architecture

The loop filter architecture with weighted feedforward summation is particularly well suited for $\Sigma\Delta$PLL design. The general architecture of this filter is shown in Fig. 6-4.

![Weighted Feedforward Loop Filter]

Since the filter uses feedforward paths to implement zeros necessary for loop stability, it avoids multiple feedback paths. In this architecture the only feedback path is through the counter. The counter converts a digital value, $N$, to an analog time, $N \times T_{\text{ref}}$. Hence it acts as a digital to analog (D/A) converter. If multiple feedback loops were used in the $\Sigma\Delta$PLL, an additional D/A converter would be required to convert $N$ into an analog voltage. The feedforward topology avoids this. However this topology does have one complication. The path, $FF_1$, requires that the current pulse from the phase detector is multiplied by $A$ and summed with the other feedforward paths (Fig. 6-4). While this is possible, it is inconvenient since the outputs of the other paths are voltages that remain

*Fig. 6-4: Weighted feedforward loop filter.*
constant during the sampling period. One way to solve this problem is to remove the feedfoward path, FF1. The zero added to the transfer function by this path must be replaced to ensure loop stability. A lead compensator (LC) can be used to replace this zero [49]. The adjusted loop filter with the lead compensator is shown in Fig. 6-5.

![Adjusted Loop Filter](image)

*Fig. 6-5: Loop filter with path FF1 removed and replaced a lead compensator.*

It has been shown that lead compensation can also be employed in the feedback loop to obtain good loop stability [50]. Since the compensation is in the feedback loop it can be implemented digitally. Thus avoiding the necessity of implementing the block with discrete time analog circuitry. In addition, when digital compensation is used (i.e. \( D(z) = 2 - z^{-1} \)), it does not affect the signal transfer function because \( D(z) \approx 1 \) in the desired signal bandwidth, BW. The resulting \( \Sigma \Delta \) PLL architecture is shown in Fig. 6-6 where \( G(z) \) determines the order of \( \Sigma \Delta \) PLL. This filter architecture is similar to a typical PLL. In a typical PLL an integrator directly follows the phase detector. Then depending on the
order of the PLL, the output of the integrator is filtered further before it is fed back to the voltage-controlled oscillator.

![Diagram of Sigma Delta PLL with digital lead compensator.](image)

*Fig. 6-6: Sigma Delta PLL with digital lead compensator.*

The loop filter, $G(z)$, in Fig. 6-6 can be designed to implement different order $\Sigma\Delta$PLLs and hence noise transfer functions (NTFs). For instance, a 2$^{nd}$ order $\Sigma\Delta$PLL can be designed by setting $G(z) = 1$. The topologies for $G(z)$ that implement third and forth order $\Sigma\Delta$PLLs are shown in Fig. 6-7.
6.1 Sigma Delta PLL Period Measurement System

The linear model presented in this section can be used to analyze the stability of a ΣΔPLL for a given loop filter with the techniques developed in [52]. However to evaluate the overall performance of the ΣΔPLL a nonlinear model must be developed.

6.1.3 ΣΔPLL A/D Size and Loop Order

In chapter 4 the DETF oscillator was determined to have a resolution of 18 nε in a 10kHz bandwidth. If the ΣΔPLL contributes an equal amount of noise as the DETF oscillator the overall resolution of the strain measurement system will be approximately 26 nano-strain (nε) which is well within the 100 nε specification. Therefore 18 nε is a suitable target resolution for the ΣΔPLL prototype. Once a resolution goal is established, the required A/D size and loop order of the ΣΔPLL can be determined.
The size of the A/D is related to the total range of the DETF oscillator period, $T_{in}$, and the size of the reference clock period. The approximate number of levels, $N_{levels}$, required by the A/D converter is given by (6-5). The resulting bits required is, $N_{bits} = \log_2(N_{levels})$.

$$N_{levels} \approx \text{Ceil} \left[ \frac{T_{max} - T_{min}}{2T_{ref}} + 3 \right]$$  \hspace{1cm} (6-5)

$$T_{max} = \frac{1}{f_r - \frac{\partial f_r}{\partial \varepsilon} |\varepsilon_{max}|}$$  \hspace{1cm} (6-6)

$$T_{min} = \frac{1}{f_r + \frac{\partial f_r}{\partial \varepsilon} |\varepsilon_{max}|}$$  \hspace{1cm} (6-7)

where $\varepsilon_{max}$ is maximum input strain. For the strain sensor in this work $\varepsilon_{max} = \pm 1000 \mu\varepsilon$, $f_r = 217$ kHz, and $\frac{\partial f_r}{\partial \varepsilon} = 39$ Hz/$\mu\varepsilon$. A plot of (6-5) is shown in Fig. 6-8.

---

*Fig. 6-8: A/D converter size versus reference frequency.*
The quantization error made by the ΣΔPLL is proportional to the period $T_{\text{ref}}$. As quantization noise grows the order of the ΣΔPLL must increase to obtain the same resolution in a given bandwidth. As a result, A/D size ($N_{\text{bits}}$) decreases with increasing ΣΔPLL loop order for a given resolution. A nonlinear model of the ΣΔPLL can be used to determine this relationship (Fig. 6-9).

The model shown in Fig. 6-9 includes the actual nonlinear model for the A/D converter instead of the linear model (Fig. 6-6). The model (Fig. 6-9) was simulated using Matlab™ resulting in Fig. 6-10 with $K_d \times K_1 = f_{\text{ref}} / A_Q$.

Note the A/D converter gain, $A_Q$, is approximately equal to $1/\text{LSB}$, where the LSB is the quantization step size of the converter. The values of $G(z)$ used in the simulation for the third and fourth order ΣΔPLL loops were:

$$G(z)\bigg|_{\text{3rd}} = 0.898 \left\{ \frac{1 - 0.795 z^{-1}}{1 - z^{-1}} \right\}$$

(6-8)
From Fig. 6-10 it can be seen that the 4th order SDPLL achieves the best resolution for a given A/D size (N_{bits}). Also the 4th order sigma delta achieves an effective resolution of 22 ne with an A/D converter size of only 4 bits. This is very close to the target resolution for the \( \Sigma \Delta \)PLL discussed earlier. Hence this was the order and A/D size used for the \( \Sigma \Delta \)PLL prototype. A plot of the noise transfer function (NTF) of the \( \Sigma \Delta \)PLL with \( G(z) \)
given by (6-9) for both the linear (Fig. 6-6) and non-linear model (Fig. 6-9) is shown in Fig. 6-11.

6.2 Prototype Sigma Delta PLL

The prototype Σ∆PLL used in this work was implemented on a PC board. The control logic, counter, phase detector, and digital lead compensation were implemented using and field programmable logic array (FPGA) (Fig. 6-12). The loop filter in this prototype was designed with continuous-time (C-T) circuitry. Finally a 4 bit flash A/D converter was used to quantize the output of the analog filter.

*Fig. 6-11: Noise transfer function for the 4th order ΣΔPLL.*
6.2 Prototype Sigma Delta PLL

6.2.1 FPGA design

6.2.1.1 Timing Control Logic

One of the most important parts of the design is the timing control (Fig. 6-12) for the \( \Sigma \Delta \) PLL. The timing control logic must generate control signals that determine when the A/D samples the output of the loop filter and when the FPGA latches the A/D output.

For this prototype an analog devices AD75482AST A/D converter was used [53]. Fig. 6-13 shows the timing control signals generated by the FPGA and the A/D. From Fig. 6-13 it can be seen that the A/D converter must convert the output of the loop filter, \( V_{LF} \), at a time greater than \( \max(t_n, t_{\tau_n}) \). To do so, the timing control circuitry generates a signal, CONVSTB, which starts an analog to digital conversion on its falling edge. The logic
circuit used to generate CONVSTB is shown in Fig. 6-14. Muller c-elements [54] are used to create an output signal that switches after its last input signal has switched ($V_{out}$ or $C_{out}$). The rising edge of the c-element output is then used to generate a pulse. This function is performed for both edges of the input signals (rising and falling), thereby generating a pulse that occurs after the last zero crossing ($\max(t_n, \tau_n)$) of the input signals.

In addition, the CONVSTB is delayed by, $t_d$, to ensure that the output of the integrator has settled before the analog to digital conversion is performed. The pulse width and delay, $t_d$, of CONVSTB are equal to multiples of the reference clock period. The BUSYB signal is a handshake signal generated by the A/D. The rising edge of the BUSYB signal indicates when the conversion is finished. Finally the rising edge of INTCLKB is used to

![Fig. 6-13: Timing control signals.](image)
latch the data from the converter and to provide a clock for the digital compensator. Finally, note the total conversion time, $t_c$, must be limited to a fraction of the minimum expected time between $t_n$ and $t_{n+1}$.

![Logic circuit used to generate CONVSTB.](image)

Fig. 6-14: Logic circuit used to generate CONVSTB.

### 6.2.1.2 Phase Detector

The phase detector used in the FPGA design is shown in Fig. 6-15. It uses two three state phase detectors [55]. One phase detector is used for detecting the difference between the zero crossings of $V_{out}$ and $C_{out}$ on their positive edge transitions and the other for negative edge transitions. The output of the phase detectors control tri-state buffers which drive the first stage of the loop filter (integrator). The output pulse of the phase detector has an area of $A_n$ (Fig. 6-2). For this phase detector, $A_n$ is given by (6-10).
where $V_{dd}$ is the tri-state buffer power supply and $R$ is the “on” resistance of the buffers. The integrator is assumed to have a DC bias of $V_{dd}/2$ at its input. Hence the output current of the tri-state buffers has three states: 0 and $\pm V_{dd}/2R$. Therefore the gain of the phase detector, $K_d$, is $V_{dd}/2R$.

Fig. 6-15: $\Sigma\Delta$PLL prototype phase detector.
6.2 Prototype Sigma Delta PLL

6.2.1.3 Counter

The counter used in the prototype is shown in Fig. 6-16. It consists of a reset-able counter and a digital comparator. The counter’s input is \( K + M \), i.e. the output of the digital compensator \( K \) plus a constant offset \( M \). When the counter reaches a value greater than its input, the digital comparator resets the counter and toggles the output \( C_{out} \).

\[
M = \text{floor} \left( \frac{f_{ref}}{2f_{osc}} \right)
\]

![Counter block diagram](image)

Fig. 6-16: Counter block diagram.

The relationship between the input of the counter and period of \( C_{out} \) is given by (6-11).

\[
\frac{T_n}{2} = \frac{T_{ref}}{2} (K_n + M + 1) \quad (6-11)
\]

6.2.2 Continuous-Time Filter Design

This section describes the design of the loop filter. It was shown the 4\(^{th}\) order \( \Sigma \Delta \text{PLL} \) with the transfer function given by (6-9) meets the resolution requirements for the resonant strain sensor. The filter in this prototype is a continuous-time (C-T) filter, while
the filter developed in [56] is a discrete-time (D-T) filter. As a result discrete-time filter, G(z), must be converted to an equivalent continuous-time filter, G(s). Two properties of the \( \Sigma\Delta \)PLL are used to perform this conversion.

First, the C-T integrator in the loop filter is functionally equivalent (Fig. 6-17a) to a discrete-time integrator followed by a zero order hold (ZOH). The outputs of the two integrators differ only during the time when the difference, \( t_n - \tau_n \), is sampled. Therefore the C-T integrator can be replaced with a D-T integrator and a ZOH (Fig. 6-17b).

![Fig. 6-17: C-T integrator and D-T equivalent.](image-url)
Second, the output of $G(s)$ is sampled and held by the A/D converter. As a result, the section label “Equivalent Digital Filter” in Fig. 6-18a can be converted to an equivalent discrete time filter $G(z)$ using state space methods detailed in [57].

The Matlab™ functions `c2d` (continuous to discrete) and `d2c` (discrete to continuous) implement the method described in [57] and were used to perform the conversion. Hence, given (6-9), the equivalent $G(s)$ can be obtained by using `d2c` with the ZOH conversion method (6-12).

$$
G(s) \bigg|_{4\text{th}} = 0.541 \left[ \frac{s^2 + 5.48 \times 10^4 s + 1.67 \times 10^{10}}{s^2 + 1934s + 4.33 \times 10^9} \right]
$$

Fig. 6-18: a) Loop filter with C-T filter $G(s)$. b) Loop filter with equivalent D-T time filter $G(z)$. 

The Matlab™ functions `c2d` (continuous to discrete) and `d2c` (discrete to continuous) implement the method described in [57] and were used to perform the conversion. Hence, given (6-9), the equivalent $G(s)$ can be obtained by using `d2c` with the ZOH conversion method (6-12).
The transfer function in (6-12) can be implemented using the block diagram shown in Fig. 6-19. The parameters \(G_1\), \(G_2\), \(G_T\) and \(FF_1\) of the filter in Fig. 6-19 are obtained by setting \(FF_2\) and \(\gamma\) to \(-1\), and calculating its transfer function, \(G_c(s)\). \(G_1\), \(G_2\), \(G_T\) and \(FF_1\) are then determined by equating the coefficients of \(G_c(s)\) to the coefficients in (6-12). The resulting parameters are given in Table 6-1.

**Table 6-1: Filter parameters for \(G_c(s)\)**

<table>
<thead>
<tr>
<th>Filter Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(FF_1)</td>
<td>0.3642</td>
</tr>
<tr>
<td>(FF_2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(G_1)</td>
<td>(-1.9249 \times 10^4) Hz</td>
</tr>
<tr>
<td>(G_2)</td>
<td>(-2.2497 \times 10^4) Hz</td>
</tr>
<tr>
<td>(p)</td>
<td>966.7595</td>
</tr>
<tr>
<td>(G_T)</td>
<td>1.4865</td>
</tr>
</tbody>
</table>
In section 6.1.3, \( G(z) \) was designed assuming \( K_d \times K_1 = f_{\text{ref}}/A_Q \). Hence the integrator gain for the filter is given by (6-13). The values of all other parameter for the prototypes loop filter are given in Table 6-2.

\[
K_1 = \frac{1}{C_{\text{int}}} = \frac{f_{\text{ref}}}{A_Q V_{\text{dd}}} \quad \text{(6-13)}
\]

**Table 6-2: Other filter parameters for \( G(s) \).**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{\text{ref}} )</td>
<td>Reference Clock</td>
<td>16 MHz</td>
</tr>
<tr>
<td>( R )</td>
<td>Phase Detector Resistance</td>
<td>180 ( \Omega )</td>
</tr>
<tr>
<td>( A_Q )</td>
<td>( \frac{2^{N_{\text{bits}}}-1}{2V_{\text{FS}}(1-2^{-(N_{\text{bits}}+1)})} )</td>
<td>A/D converter Gain 6.1935 1/V</td>
</tr>
<tr>
<td>( V_{\text{dd}} )</td>
<td>Phase Detector Supply voltage</td>
<td>3.3 V</td>
</tr>
<tr>
<td>( N_{\text{bits}} )</td>
<td># of A/D bits</td>
<td>4</td>
</tr>
<tr>
<td>( V_{\text{FS}} )</td>
<td>Full scale voltage of A/D, A/D input range = 0 V to 2( V_{\text{FS}} )</td>
<td>1.25 V</td>
</tr>
<tr>
<td>( C_{\text{int}} = 1/ K_1 )</td>
<td>Calculated integrator capacitance</td>
<td>3.55 nF</td>
</tr>
</tbody>
</table>

Finally the circuit that was used to implement the loop filter is shown in Fig. 6-20. Note that the inversion at the output of the filter can be removed by changing the polarity of the phase detector. This is accomplished by reversing the phase detector inputs. For the prototype designed in this work, the last inverting stage was not removed but was used as part of the level shifting necessary to place the filters DC output in the middle of the A/D input range.
The PC board implementation of the full $\Sigma\Delta$PLL from (Fig. 6-12) is shown in Fig. 6-21. The full $\Sigma\Delta$PLL consists of two boards, one for the FPGA and one containing the analog filter $G(s)$, and the A/D. The FPGA board used was a Xilinx Virtex –E PQ240-110 development board [58]. A 12-bit A/D (AD7482) was used so that the number of bits used by the $\Sigma\Delta$PLL was programmable through reconfiguration of the FPGA. For the results presented in the following sections the prototype used the four most significant bits of the 12-bit A/D output.
6.3 Measured Results

6.3.1 Experimental Setup

The experimental setup used for testing the prototype $\Sigma \Delta$PLL is shown in Fig. 6-22.

Fig. 6-22: Experimental Setup.
In the experimental setup the ΣΔPLL could be used to measure the output of the DETF oscillator or the output of the signal generator #1. The output of signal generator #1 was used to measure the noise transfer function (NTF) of the ΣΔPLL (Fig. 6-23). It was observed that the measured NTF did not match the simulated NTF. Instead of having the expected slope of 40 dB per decade, the measured NTF had a slope of 20 dB per decade (Fig. 6-23). Further experiments revealed that a mismatch in the phase detector (PD) was the source of this discrepancy. The PD was implemented using standard FPGA tri-state output drivers and the mismatch in the driver’s pull-up and pull-down strength was measured to be 2 percent. Once the mismatch was incorporated into the simulation model the simulated and measured results were in good agreement.

Fig. 6-23: Measured and simulated noise transfer functions for the prototype ΣΔPLL.
The output of the DETF oscillator with zero input strain was then measured with the ΣΔPLL and compared to the simulation model. The simulation model incorporated the PD mismatch and the measured phase noise of oscillator (see chapt. 4). The simulation model predicted a 29 nε resolution in 10 kHz bandwidth. The actual measurement achieved a resolution of 33 nε in 10 kHz (Fig. 6-24). The measured noise floor is approximately 60 pε/√Hz or 12 fm/√Hz up to 1 kHz.

The measured resolution, due to the resonant sensor oscillator only was 18nε. Therefore the ΣΔPLL contributes approximately 24nε to the overall strain measurement system resolution.

![Fig. 6-24: Measured and simulated sensor noise floor.](image)
Finally, the measurement system was characterized by applying strain at varying frequencies using the strain actuator. Using this method the operation up to 10 kHz bandwidth was verified (Fig. 6-25).

6.4 Discussion

A complete resonant strain measurement system that obtains 33 nε in a 10 kHz bandwidth has been developed. A novel 4th order ΣΔPLL was used to demodulate and digitize the output of the resonant sensor oscillator reported in [2]. In addition this system demonstrates that using a ΣΔPLL for resonant sensing has the potential to achieve high measurement resolution, while replacing the PLL and A/D converter with a signal block, thus reducing the complexity of the frequency demodulation electronics.

Fig. 6-25: Measured strain applied varying frequencies.
CHAPTER 7:
CONCLUSIONS AND FUTURE WORK

7.1 Summary

In the past attempts to improve the resolution of MEMS double-ended tuning fork (DETF) resonant sensors have concentrated on reducing the resonator phase noise. This was based on the hypothesis that making the resonator Q as large as possible, phase noise can be made arbitrarily small. If true, this would imply that there is no resolution limit for MEMS resonant sensors. However, further analysis of sensor phase noise reveals a more complex relationship between phase noise, resonator drive level and Q. In fact this relationship shows that increasing Q increases phase noise in many cases. In this thesis we present a more accurate model for resonator phase noise that accounts for both Q and drive level.

The main results of this analysis show that phase noise does not improve with Q in all cases. In fact “far from the carrier noise” phase noise, $L_{\text{far}}$, gets worse with increasing Q, while “close to the carrier noise” phase noise, $L_{\text{near}}$, improves. This general result is derived in chapter 3 and repeated here (7-1). Note, $x_b$ is the peak deflection of the resonator.
Resolution is proportional to the integral of the sensors phase noise over the bandwidth of interest. Therefore, it can be shown, from (7-1), that there is a Q for which resolution is optimized over that bandwidth. This optimum Q was shown to be approximately equal to (7-2) (see chapter 3).

\[
Q_{\text{opt}} = \frac{\sqrt{3} f_r}{2 \text{ BW}} \quad (7-2)
\]

where \( f_r \) is the resonant frequency of the sensor and BW is the sensor bandwidth. Hence as sensor bandwidth increases the Q at which optimum resolution is attained decreases. This result represents a radical shift in how Q is selected for resonant sensor design.

Consider a sensor that has a resonant frequency of 200 kHz and a required bandwidth of 1 kHz. The optimum Q for this sensor would only be \( \sim 173! \) A resonant strain sensor oscillator was developed in this work (chapter 4) that demonstrates the improvement in resolution and phase noise (L\text{far}) when operating close to the optimum Q. It obtained a phase noise floor (L\text{far}) that was \( \sim 10000 \) times smaller than previous resonant sensors with similar resonant frequencies but with Q’s greater than 10000.

One of the challenges for operation at low Q is the presence of parasitic feed-through capacitance, \( C_{\text{ft}} \). \( C_{\text{ft}} \) adds an undesirable current to the total output current of the resonant sensor and very small values of this capacitance can completely swamp out the DETF
output. There is a direct relationship between Q and the value of $C_{ft}$ that will make oscillation, and hence detection of the resonator output, impossible. This relationship derived in chapter 4 is repeated here (7-3).

$$\max(C_{ft}) = \frac{Q}{2} \cdot C_x = \frac{1}{4 \pi f_0 R_x}$$

$$Q_{\min} = 2 \frac{C_{ft}}{C_x}$$

Here $C_x$ and $R_x$ are the motional capacitance and resistance in the resonator linear model (see chapter 2). For robust oscillator design we require $C_{ft} \leq \frac{\max(C_{ft})}{5}$ or $Q \geq 5 \times Q_{\min}$. To meet this condition at atmospheric pressure with the sensor developed in this work ($Q=370$), $C_{ft}$ would have to be less than $4\text{fF}$! As a result, the typical solution is to increase $Q$ until $Q \gg 5 \times Q_{\min}$ by operating the sensor in a vacuum. However, by increasing $Q$ to ensure oscillation, resolution is adversely affected.

The goal of this work was to operate at the optimum $Q$ of the sensor to obtain the predicted improvement in phase noise and thus resolution. However in many cases the optimum $Q$ ($Q_{\text{opt}}$) can be much lower than the $Q$ required to overcome the feed-through capacitance. Therefore, $C_{ft}$ can become the limiting factor with respect to phase noise and resolution. In chapter 4, a time variant oscillator, or square wave drive oscillator (SWO), was introduced which alleviates this problem by allowing oscillation of low $Q$ resonators in the presence of large feed-through capacitance. This work demonstrated oscillation of a DETF resonant sensor with a $Q = 370$ (closer to its optimum $Q$ of 17), and with a feed-through capacitance of $\sim 70\text{fF}$ using the SWO oscillator. This resonant sensor oscillator
achieved the best phase noise floor to date for resonant sensors in this resonant frequency range.

Finally, a 4th order sigma-delta phase locked loop (Σ∆PLL) frequency measurement system was developed (chapters 5 and 6) to measure and digitize the output frequency of the SWO oscillator mentioned above. This Σ∆PLL replaced the typical method used to measure the output of a resonant sensor oscillator. The old method employed a two-step approach where a phase locked-loop (PLL) is used to demodulate the oscillator output, followed by a digitization step done by an A/D converter. One of the main problems with this method was that the voltage-controlled oscillators (VCOs) used in the PLL feedback loop adversely affects the performance of the overall system. In addition, this method does not take advantage of the inherent oversampling available with resonant sensors (chapter 5). As a result this method will require a more complex A/D converter than the Σ∆PLL system. The Σ∆PLL combines the two steps (demodulation and digitization) into a single step which simplifies the design of the measurement system. In addition it takes advantage of oversampling allowing simplification of the A/D converter design. Finally, the Σ∆PLL uses a digital counter instead of the VCO. The counter accuracy is dependent on the frequency reference used to drive it. With a high accuracy reference, such as an external crystal oscillator, the variation in the system due to the counter is much lower than the variation introduced by a VCO. The Σ∆PLL developed in this work was the first 4th order Σ∆PLL published. In addition a novel method was developed to double the sample rate of the Σ∆PLL by sampling on both edges of the input signal.
The resonant sensor measurement system with a SWO oscillator and a ΣΔPLL was implemented with surface mount components on a PC board to demonstrate the feasibility of these concepts. With this SWO oscillator, we obtained a phase noise floor of $-120$ dBC/Hz. This is the best noise performance obtained to date for resonant sensors in this resonant frequency range. This confirmed the conclusions from the model that predicted an improved phase noise floor as the Q is reduced. The 4th order ΣΔPLL implementation achieved a 1Hz frequency resolution (21 ps period resolution) in a 10kHz bandwidth. This overall measurement system was used to implement a prototype strain sensor and achieved a 33 nε resolution in 10kHz.

7.2 Future Directions

7.2.1 Design of Better MEMS Resonators (Phase Noise)

Interestingly, one of the ways to improve the performance of the sensor presented in this work could be improved would be to use a sinusoidal oscillator instead of the SWO oscillator. This is because the nonlinearity of the SWO oscillator causes the phase noise to be larger than the phase noise of a linear or sinusoidal-output oscillator. Making this improvement with the DETF resonant sensor presented in this work was impossible because $C_{fr}$ had to be less than 4fF to use a linear oscillator. However if the sensor itself can be improved, this limitation could be removed. For instance, the sensor used in this work had gaps ~3µm. If the gaps were decreased than by a factor of $\nu$ then the minimum Q required for oscillation would decrease by a factor of $\nu^2$ or the value of $C_{fr}$ at which oscillation would become impossible for sinusoidal-output oscillators would increase by
Hence a sinusoidal oscillator could be used and the phase noise of the sensor could be improved.

In addition to decreasing the gaps to allow for oscillation with large $C_{in}$, feed-through capacitance can also be reduced by implementing the sensor with integrated electronics and a differential sensor topology! Finally improving the mechanical linearity of the DETF resonator would improve the long-term stability of the resonant sensor oscillator.

### 7.2.2 Temperature Compensation

The strain sensor implemented in this work exhibited a temperature sensitivity of approximately 7 HzºC. As a result a 1 degree (Celsius) change in temperature will result in an apparent strain of 0.18 microstrain which is larger than the resolution goal for the sensor is 0.1 microstrain. Therefore it is obvious that some sort of temperature compensation will be required to obtain the resolution goal. In the work presented here temperature compensation for the resonant strain sensor was not employed. However one of the compensation schemes outlined in chapter 2 could be pursued in the future to improve sensor performance.

### 7.2.3 Sigma Delta Phase Locked Loop: Other Applications

The sigma delta phase locked loop has applications in other areas. One possible application is a low power digital FM radio. Presently digital radios are implemented using the architecture shown in Fig. 7-1. This type of FM radio operates by shifting one of the FM radio channels in the band (88 MHz to 108MHz) to an intermediate frequency (IF). Typically the IF frequency is 10.7MHz, but can be lower. A ceramic IF filter with a
10.7MHz center frequency and a 200kHz bandwidth is used to select the desired channel while attenuating all other channels in the FM band.

**Fig. 7-1: State of the art Digital AM/FM radio architecture.**

The output of the filter is then digitized with a band pass sigma delta A/D converter. Finally the FM channel is demodulated digitally. One of the advantages of this architecture is that both FM and AM radio can be demodulated using the same front-end receiver architecture. In addition there is some evidence that the ceramic IF filter may not be needed as the filter in the band pass sigma delta can be used to perform its function [59]. While this type of architecture provides flexibility it requires a complicated band pass sigma delta A/D converter to convert the FM radio band [59, 60, 61]. Typically the A/D requires a bandwidth in excess of 200kHz, and an SNR greater than 70dB to achieve a SNR around 66dB [53] at the output of the demodulator. This is because digital demodulation introduces quantization noise that further degrades the overall SNR [62]. In addition these converters [59, 60, 61] have sample rates in the range of 20-80MHz depending on the IF frequency used. An alternative to this architecture is shown in Fig. 7-2. Here a ΣΔPLL is used to demodulate and digitize the FM radio channel. This method
cannot implement AM demodulation, unlike the previously described method. However, it is possible to achieve similar or better performance than reported in [59, 60, 61] for FM demodulation and digitization with sample rates as low as 1MHz.

The main reason such a low sample rate can be used, is because the \( \Sigma \Delta \) PLL only needs to digitize the signal bandwidth, which is approximately 20kHz for FM radio. The method shown in Fig. 7-1 must digitize the entire FM modulated radio channel bandwidth, which is approximately 190 kHz, centered at the IF frequency, assuming a 75 kHz frequency deviation [41]. Hence the A/D converter must operate at much higher sample rates. To demonstrate the feasibility of the \( \Sigma \Delta \) PLL method, the PC board prototype was tested as an FM demodulator in the configuration shown in Fig. 7-3. An FM modulated 10.7MHz signal was fed into the counter of the \( \Sigma \Delta \) PLL and the sample rate was 2×217 kHz or 434 kHz. In this experiment the \( \Sigma \Delta \) PLL achieved better than 94dB SNR in 10 kHz (Fig. 7-4).

**Fig. 7-2: Low power \( \Sigma \Delta \) PLL FM radio.**
7.2 Future Directions

Fig. 7-3: Test setup for a FM radio using $\Sigma\Delta$PLL prototype.

Fig. 7-4: Measured results: demodulated 10.7MHz IF radio signal for frequency deviations of 200Hz and 75kHz at a modulation frequency of 1kHz.
Assuming the bandwidth is increased to 20kHz, and the oversampling ratio remains the same, this architecture should achieve a similar SNR in 20kHz at a sample rate of 868 kHz. This is a reduction of 20 to 80 times the sample rate used in [59, 60, 61]! Therefore a very low power digital FM radio could be achieved using the \(\Sigma\DeltaPLL\) architecture shown in Fig. 7-2.

Finally this architecture (Fig. 7-3) could be used with high frequency resonant sensors. In this case, the resonant sensor would drive the input of the counter (Fig. 7-5).

\[ N_{out} \approx \frac{f_{sensor}}{2 f_s} \]

\(f_{sensor}\)

**Fig. 7-5**: SDPLL architecture for measurement of high frequency resonant sensors.
Resonator nonlinearities can result in significant $1/f^3$ phase noise. This appendix derives an analytical model for this noise. Consider the oscillator in Fig. A.1. For this analysis we assume all $1/f$ noise is referred to the input of the resonator ($n_d$).

Assuming we have an electro-statically driven resonator with equal drive and sense capacitances the steady state driving force, $F_{\text{drive}}$, applied to the resonator is:

$$F_{\text{drive}} = -\frac{1}{2} \frac{\partial C}{\partial x} \left( V_{\text{out}} + v_{1/f} - V_{dc} - V_{dc}^2 \right)$$  \hspace{1cm} (A-1)
Appendix A

\[ V_{\text{out}} = v_d \cos(\omega_o t) \]  

(A-2)

Substituting (A-2) into (A-1) and taking terms only at \( \omega_o \) the driving force reduce to:

\[ F_{\text{drive}} = \frac{\partial C}{\partial x} (V_{dc} - v_{1/f}) v_d \cos(\omega_o t) = F_d \cos(\omega_o t) \]  

(A-3)

Note that the electrostatic nonlinearity has up-converted the 1/f noise to the resonant frequency, resulting in a peak driving force that is amplitude modulated by the 1/f noise. As a result the peak deflection, \( x_b \), of the resonator is amplitude modulated by 1/f noise. It can be shown that the peak deflection of a resonator as function of the peak driving force, \( F_d \) is:

\[ \frac{x_b}{F_d} = \frac{Q}{k_{ml}} \]  

(A-4)

It was shown in chapter 2 that the resonant frequency is a function of the peak amplitude when a 3rd order mechanical or electrical nonlinearity is present. As a result the frequency noise due to the 1/f modulation can be found by noting:

\[ \Delta f^2 = \left( \frac{3}{8} f_o \frac{k_3}{k_{ml}} x_b^2 \right)^2 \]  

(A-5)

Plugging (A-3) into (A-5) and taking only the terms that are linear with respect to the 1/f noise we get:

\[ \Delta f^2 = \left( \frac{9}{16} f_o^2 k_3^2 Q^4 V_{dc}^2 v_d^4 \right) \v_{1/f}^2 = \left( \frac{9}{16} f_o^2 k_3^2 \frac{x_b^4}{V_{dc}^2} \right) \v_{1/f}^2 \]  

(A-6)

Note that:
\[ v_{\text{rf}}^2 = \frac{K_f}{f_m} \]  \hspace{1cm} (A-7)

And phase noise is related to frequency noise by:

\[ S_\phi(f_m) = \frac{\Delta f^2}{f_m^2} = \left( \frac{9}{16} \frac{f_0^2}{f_m^2} \frac{k_B^2}{k_{\text{ml}}} Q^4 V_{\text{dc}}^2 V_d^4 \right) \frac{K_f}{f_m^3} = \left( \frac{9}{16} \frac{f_0^2}{f_m^2} \frac{k_B^2}{k_{\text{ml}}} V_{\text{dc}}^4 \right) \frac{K_f}{f_m^3} \]  \hspace{1cm} (A-8)

To confirm this model, simulations were done with Matlab™. The results of a Matlab™ simulation modeling the DETF resonator from chapter 4 are shown in Fig. A.1. Only 1/f noise was present in the oscillator simulation.

![Fig. A.2: 1/f^3 phase noise due to resonator nonlinearities.](image)
Appendix A

The parameters for the simulation were:

Table A-1: Simulation Parameters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_d$</td>
<td>4.7 V</td>
</tr>
<tr>
<td>$V_{dc}$</td>
<td>84 V</td>
</tr>
<tr>
<td>$f_0$</td>
<td>217 kHz</td>
</tr>
<tr>
<td>$Q$</td>
<td>270</td>
</tr>
<tr>
<td>$K_f$</td>
<td>$6.49 \times 10^{-12}$ V²</td>
</tr>
<tr>
<td>$k_{m1}$</td>
<td>658.3 N/m</td>
</tr>
<tr>
<td>$k_3$</td>
<td>$7.2 \times 10^{12}$ N/m³</td>
</tr>
</tbody>
</table>

It was observed that as the drive voltage was increased, the phase noise displayed a region of $1/f^4$ or $1/f^5$ phase noise that occurred after the 3dB bandwidth ($f_0/2Q$) of the oscillator loop. Our hypothesis for this behavior [63] was that this could be a result of the peak deflection being a frequency dependent function of the peak force. In fact it is related to the peak deflection by [1, 26]:

$$\frac{x_b}{F_d} = \frac{Q}{k_{m1}} \frac{1}{2Q + 1} \frac{1}{\omega_0 + 1}$$  \hspace{1cm} (A-9)

If this is taken into account, the phase noise due to the 3rd order nonlinearity is can be approximated by the empirical equation below:
Figures A-3 and A-4 show simulations where the drive voltage was increased to 9.1V and 28.8V respectively while the other parameters remain unchanged (table A-1). Note the presence of the $1/f^5$ phase noise.

A zero is also added into the phase noise transfer function by “VGA” feedback loop used to control the peak deflection of the oscillator [63] indicated in Fig. A.3. Interestingly, the position of the zero is dependent peak drive voltage, $v_d$. For instance, the zero moves to higher in frequency (Fig. A.4) as the drive voltage is increased from 10V to 30V. Because of this zero, it is difficult to verify the presence of $1/f^5$ phase noise in Fig. A.3.
The above analysis shows that 3rd order nonlinearity does in fact cause $1/f^3$ phase noise. In addition it can cause higher order noise phase noise as well. Therefore nonlinearities should be minimized to reduce this type of phase noise. Finally, equations (A-9) and (A-10) approximate the phase noise in due to these nonlinearities however more research needs to be focused on developing better analytical models for nonlinear mechanisms in MEMS resonators. This is particularly evident from (Fig. A.3 and Fig. A.4) where the simulation results do not match the predicted results from equation (A-10). In fact it seems that the phase noise may be exhibiting $1/f^4$ noise, which is not predicted by the model. However the equations do provide an upper bound for the noise.
APPENDIX B:

SWO OSCILLATOR SCHEMATIC AND PC BOARD

Fig. B.1 PC Board SWO Oscillator.

The comparator for the SWO (Fig. B.2) is implemented with the amplifiers A1, A2, and A3. A3 generates an inverted signal, DRIVEB, which can be used to drive a capacitor, C_{neg}, in parallel with the DETF. This provides a method to cancel C_{ft} by adjusting the values of RP_1 and RP_2. Finally RP_3 can be used to tune the frequency at which the phase shift of the circuit is zero. This is equivalent to adjusting the SWO oscillator frequency when the resonator bias is zero, to a value larger than the resonators frequency. Note that the schematic in Fig. B.2 is only a partial schematic for the PC board shown above but is sufficient to build a similar board.
Fig. B.2: Schematic of SWO PC Board Oscillator.
BIBLIOGRAPHY


[63] *Personal Communication with Baris Cagdaser*. 